Problem 1. (12 pts.) (a) We have $\sigma = 5$, $n = 16$, $\bar{x} = 20$. The $z$-confidence interval for the mean is $\bar{x} \pm z_{\alpha/2} \sigma / \sqrt{n}$. So the 95% confidence interval is approximately

$$20 \pm 1.96 \cdot 5/4 = 20 \pm 2.45 = [17.55, 22.45].$$

(b) 50% confidence interval: $\alpha = 0.5$, so $z_{\alpha/2} \approx 0.674$. So the 50% confidence interval is approximately

$$20 \pm 0.6745 \cdot 5/4 = 20 \pm 0.843 = [19.157, 20.843].$$

Likewise the 99% confidence interval is approximately

$$20 \pm 2.576 \cdot 5/4 = 20 \pm 3.22 = [16.780, 23.220].$$

(c) The formula for the confidence interval has width $2z_{\alpha/2}\sigma/\sqrt{n}$. Since $\sigma$ and $z_{\alpha/2}$ are fixed, this gets smaller as $n$ increases.

(d) The width of the interval is $2 \cdot 1.96 \cdot 5/\sqrt{n}$. A little arithmetic gives.

$$\frac{2 \cdot 1.96 \cdot 5}{\sqrt{n}} < 1 \iff 19.6 < \sqrt{n} \iff 384.16 < n.$$

**answer:** $n = 385$.

Problem 2. (18 pts.) (a) This is a $t$-confidence interval. All computations were done in R. We have: sample mean $\bar{x} = 13.31$, sample variance $s^2 = 4.73$, sample size $n = 10$, degrees of freedom $df = 9$.

The critical values are $t_{0.975} = qt(0.025, 9) \approx -2.26$ $t_{0.025} = qt(0.975, 9) \approx 2.26$.

So the 95% confidence interval for the mean is approximately

$$13.31 \pm 2.26 \cdot s/\sqrt{10} = 13.31 \pm 1.56 = [11.75, 14.86].$$

(b) We’ve assumed that the lifespans of the individual mayflies are independent and normally distributed. The independence might not be a good assumption if the mayflies were all from the same geographic area: the weather or the amount of available food might all impact the lifespan of the flies. Also, it’s possible that the lifespans of male and female flies might follow different distribution. We’d have to do further exploration to decide on this.

(c) Since 14 is in the confidence interval we would not reject it.

(d) The interval for the standard deviation is the square root of that for the variance, i.e.

$$\left[\sqrt{(n-1)s^2/c_{0.025}}, \sqrt{(n-1)s^2/c_{0.975}}\right] \approx [1.495, 3.969].$$
(The number of degrees of freedom on the chi-square distribution is 9.) The values of the chi-square critical values were computed in R:

\[ c_{0.025} = \text{qchisq}(0.975, 9) \approx 2.700, \quad c_{0.975} = \text{qchisq}(0.025, 9) \approx 19.023 \]

(e) The width of the 95\% t-confidence interval is \( 2t_{0.025} \cdot s/\sqrt{n} \), where \( t_{0.025} \) is the critical value for the t-distribution with \( n - 1 \) degrees of freedom.

The problem said to base our computation on the sample variance \( s^2 \approx 4.727 \). So, to find the value of \( n \) we first estimated it using z-intervals. Then we searched the values of \( n \) near the estimate for the smallest one that gave a t-interval less than 1.

The first estimate was \( n = (2 \cdot s \cdot 1.96)^2 = 72.6 \). Then we computed t-intervals for \( n = 72, 73, 74, 75, 76 \) using the R code:

\[ \text{width} = (2 * s * qt(.025, (n-1)))^2/n \]

The first value of \( n \) that gave a width less than 1 was \([n = 76]\).

(f) No, \( n = 76 \) is not guaranteed to give a confidence interval of width less than 1 hour. The confidence interval is computed using the sample standard deviation \( s \). Since this is computed from the data it could be very large for any given random sample.

A non-mathematical reason not to trust the confidence interval is that it is derived using an assumption of independent data points. For simple experiment designs (mayflies collected from one region) in this case, the independence assumption sounds very unlikely to be correct.

**Problem 3.** (10 pts.) This is a problem about using a standardized statistic to compute confidence intervals. We are told that \( y_2/a \sim \text{beta}(2, n - 1) \). If \( c_{0.025} \) and \( c_{0.975} \) are the critical values for \( \text{beta}(2, n - 1) \) then this means that

\[ P \left( c_{0.975} < \frac{y_2}{a} < c_{0.025} \mid a \right) = 0.95. \]

Doing some algebra to isolate \( a \) in the middle, this becomes

\[ P \left( \frac{y_2}{c_{0.025}} < a < \frac{y_2}{c_{0.975}} \mid a \right) = 0.95. \]

If \( n = 9 \) we have

\[ c_{0.025} = \text{qbeta}(0.975, 2, 8) = 0.482, \quad c_{0.975} = \text{qbeta}(0.025, 2, 8) = 0.0281 \]

So, in this case, the 95\% confidence interval is

\[ \left[ \frac{y_2}{c_{0.025}}, \frac{y_2}{c_{0.975}} \right] = \left[ \frac{y_2}{0.482}, \frac{y_2}{0.0281} \right]. \]

**Problem 4.** (12 pts.) (a) We have \( x \sim \text{binomial}(n, \theta) \), so \( E(X) = n\theta \) and \( \text{Var}(X) = n\theta(1 - \theta) \). The conservative variance is just \( \frac{n}{4} \). So the distributions being plotted are
binomial(250, θ), \( N(250θ, 250θ(1 − θ)) \), \( N(250θ, 250/4) \).

Note, the whole range is from 0 to 250, but we only plotted the parts where the graphs were not essentially 0.

We notice that for each \( θ \) the blue dots lie very close to the green (true variance) curve. So the \( N(nθ, nθ(1 − θ)) \) distribution is quite close to the binomial\((n,θ)\) distribution for each of the values of \( θ \) considered. In fact, this is true for all \( θ \) by the Central Limit Theorem. For \( θ = 0.5 \) the conservative variance is the exact variance. For \( θ = 0.3 \) the conservative variance works well: it has smaller peak and slightly fatter tail. For \( θ = 0.1 \) the conservative approximation breaks down and is not very good.

In summary we can say two things about the conservative variance:

1. It gives good results for \( θ \) near 0.5 and breaks down for extreme values of \( θ \).
2. Since the conservative variance overestimates the variance (the conservative graphs are shorter and wider) it gives us a confidence interval that is larger than is strictly necessary. That is a nominal 95% conservative interval actually has a greater than 95% confidence level.
(b) Using the conservative variance, we know that $\bar{x}$ is approximately $N(\theta, 1/4n)$. For an 80% confidence interval, we have $\alpha = 0.2$ so

$$z_{\alpha/2} = \text{qnorm}(0.9, 0, 1) = 1.2815.$$ 

So the 80% confidence interval for $\theta$ is given by

$$\left[\bar{x} - \frac{z_{0.1}}{2\sqrt{n}}, \bar{x} + \frac{z_{0.1}}{2\sqrt{n}}\right] = [0.5195, 0.6005]$$

For the 95% confidence interval, we use the rule-of-thumb that $z_{0.025} \approx 2$. So the confidence interval is

$$\left[\bar{x} - \frac{1}{\sqrt{n}}, \bar{x} + \frac{1}{\sqrt{n}}\right] = [0.497, 0.623]$$

It’s okay to have used the exact value of $z_{0.025}$. This gives a confidence interval:

$$\left[\bar{x} - \frac{1.96}{2\sqrt{n}}, \bar{x} + \frac{1.96}{2\sqrt{n}}\right] = [0.498, 0.622]$$

(c) With prior Beta(1, 1), if observe $x$ and then the posterior is Beta($x+1$, 250+1−$x$). In our case $x = 140$. So, using R we get the 80% posterior probability interval:

prob_interval = $\text{qbeta}(0.1, 141, 111)$, $\text{qbeta}(0.9, 141, 111)$

= [0.51937, 0.5995]

This is quite close to the 80% confidence interval. Though the two intervals have very different technical meanings, we see that they are consistent (and numerically close). Both give a type of estimate of $\theta$.

Problem 5. (15 pts.) (a) answer: 95%. This is exactly the definition of confidence: the probability that a randomly generated interval will contain the value of $\theta$ that was used to generate it.

(b) (i) The given interval is [0.124, 0.516]. From the possible choices of $\theta$, i.e. 0, 0.2, 0.4, 0.6, 0.8, 1.0, this contains $\theta = 0.2$ and $\theta = 0.4$. Using the prior probability table we get

$$P(\theta = 0.2 \text{ or } \theta = 0.4) = 0.02.$$ 

(ii) We do the usual Bayesian update:
The prior is given above. The sample mean has distribution $\bar{x} \sim N(\theta, 1/100)$. (Remember 1/100 is the variance.) So, the likelihood

$$f(\bar{x}|\theta) = \frac{1}{\sqrt{2\pi}\cdot(1/10)}e^{-(\bar{x} - \theta)^2/(2/100)}.$$ 

We used $\text{dnorm}(0.32, \theta, 1/10)$ to compute the likelihood column in the Bayesian update table below. The posterior was computed as usual:
posterior = (prior \times likelihood)/\text{sum}(prior \times likelihood)

Here is the Bayesian update table.

| Hypothesis | prior $p(\theta)$ | likelihood $f(x = 0.32 | \theta)$ | Bayes numerator $\text{prior} \times \text{likelihood}$ | posterior $p(\theta | x = 0.32)$ |
|------------|-------------------|---------------------------------|--------------------------------|-----------------|
| 0.0        | 0.95              | 0.024                           | 0.026                           | 0.315           |
| 0.2        | 0.01              | 1.942                           | 0.019                           | 0.270           |
| 0.4        | 0.01              | 2.897                           | 0.029                           | 0.403           |
| 0.6        | 0.01              | 0.079                           | 0.008                           | 0.011           |
| 0.8        | 0.01              | $3.96 \times 10^{-5}$           | $3.96 \times 10^{-7}$           | $5.5 \times 10^{-6}$ |
| 1.0        | 0.01              | $3.63 \times 10^{-10}$          | $3.6 \times 10^{-12}$           | $5.055 \times 10^{-11}$ |
| 1.0        | 1.0               | 0.07182                          | 1.0                             |                  |

Now using the posterior probability column we get

$$P(\theta = 0.2 \text{ or } \theta = 0.4 | x = 0.32) \approx \boxed{0.674}.$$  

Neither prior or posterior probability is close to 0.95. This is not surprising the frequentist confidence interval makes no use of the prior so can’t tell us the probabilities that hypothetical values of $\theta$ are in any given interval. Both confidence intervals and Bayesian updating use the likelihood. This is reflected in the fact that the posterior probability just computed is much greater than the prior probability.