Problem 1.  (15 pts.) Poker hands.

(a) answer: (Reasons below.)

\[ P(\text{straight}) = 0.00392, \quad P(\text{four-flush}) = 0.01431, \quad \text{A four-flush is more likely than a straight.} \]

We create each hand by a sequence of actions and use the rule of product to count how many ways it can be done. (Critically: the number of choices available at each step is independent of the choices made in the earlier steps.)

**Straight:** There are 10 sequences that make a straight, e.g. A,2,3,4,5; 2,3,4,5,6; \ldots, 10,J,Q,K,A.

We build a straight by first choosing the sequence: 10 ways.
Then there are \(4^5\) ways to choose the exact cards for that sequence (4 ways for the first card, 4 for second etc.). But 4 of these are straight flushes, which the problem specifies do not count as straights. So, there are \(4^5 - 4\) ways to choose the 5 cards for a given sequence that give a straight and not a straight flush.

\[
\text{number of hands that are a straight} = 10 \cdot (4^5 - 4) = 10,200.
\]

To find the probability we divide by the number of five card hands.

\[
P(\text{straight}) = \frac{10200}{52^5} \approx 0.00392
\]

Note that this gives the probability only because every hand is equally likely.

**Four-flush**

We build a four-flush by first choosing a suit for 4 of the cards: 4 ways.

Then we choose four cards from that suit: \(\binom{13}{4}\).

Then we choose the last card from the suit of the same color: 13.

So, the number of hands that are a four-flush is

\[
4 \cdot \binom{13}{4} \cdot 13 = 37180.
\]

We divide this by the total number of hands to get

\[
P(\text{four-flush}) = \frac{37180}{52^5} \approx 0.01431,
\]

We see that a four-flush is far more likely than a straight.

You can calculate the probability of other poker hands using a similar strategy. The full list is here: [http://en.wikipedia.org/wiki/Poker_probability](http://en.wikipedia.org/wiki/Poker_probability)

(b) (i) Unlike in part (a) the order that the cards come in matters. There are 39 cards in the deck that are not spades. Thus, there are \(39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34\) ways
to deal the first six cards. After that, we choose one of 13 spades. Likewise, the number of ways to deal out any 7 cards is $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46$. Since all deals are equally likely, the probability is the ratio
\[
\frac{39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46} \approx 0.04529.
\]
Alternatively, one could view the order as not mattering within the first six cards, and thus calculate
\[
\frac{\binom{39}{6} \cdot 13}{\binom{52}{6} \cdot 46}
\]
Why does this give the same probability?

(ii) There are 13 ways to choose a diamond for the last card and 52 ways to choose the last card, so $P(\text{last card is a diamond}) = \frac{13}{52} = 0.25$.

Problem 2. (20 pts.) A roll of the dice.

(a) answer: (reasons below) $P(\text{twelve-sided die wins}) = \frac{60}{96} = 0.625$.

$P(\text{twelve-sided die wins at least two of three}) = \frac{350}{512} = 0.6836$.

For each of the possible values the twelve-sided die could roll, we calculate the probability that the eight-sided die rolls something lower.

The table below shows the dice matched against each other. Each entry is the probability of seeing the pair of numbers corresponding to that entry. We color the probability red in the rolls the twelve-sided dice would win.

<table>
<thead>
<tr>
<th>12-sided die</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-sided</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
</tr>
<tr>
<td>2</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
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<td>1/96</td>
</tr>
<tr>
<td>3</td>
<td>1/96</td>
<td>1/96</td>
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<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
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<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
</tr>
<tr>
<td>4</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
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<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
</tr>
<tr>
<td>5</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
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<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
</tr>
<tr>
<td>6</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
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<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
</tr>
<tr>
<td>7</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
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<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
</tr>
<tr>
<td>8</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
<td>1/96</td>
</tr>
</tbody>
</table>

Note that each of these probabilities is $1/96$ and since we count 60 of them, the total is $60/96 = 5/8$.

Note also that we could have counted the 60 ways that the twelve-sided die wins, without making a big table, in the following way:

If the twelve-sided die rolls a 12, then it wins 8 ways, i.e. if the 8-sided die rolls 1-8.
Likewise if the twelve-sided rolls an 11 then it wins 8 ways. We can make the table:

<table>
<thead>
<tr>
<th>12-sided roll</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td># of ways to win</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

Thus the total number of ways the 12-sided die wins is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 8 + 8 + 8 + 8 = 60$. 
For the best two of three, the probability of an outcome with two wins and one tie-or-loss for the 12-sided die is \( \frac{5}{8} \cdot \frac{5}{8} \cdot \frac{3}{8} = \frac{75}{512} \). There are three such outcomes, so a probability of \( 3 \cdot \frac{75}{512} = \frac{225}{512} \) for two wins. Add to this the probability \( \frac{5}{8} \cdot \frac{5}{8} \cdot \frac{5}{8} = \frac{125}{512} \) of three wins gives \( \frac{225 + 125}{512} \).

(b) answer: (reasons below) \( P(\text{pair of six-sided dice wins}) = \frac{514}{1152} \approx 0.4462 \).

Not all of the possible sums of two dice that could happen here have the same probability. First we calculate each of these.

For two six-sided dice:

<table>
<thead>
<tr>
<th>Sum</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/36</td>
</tr>
<tr>
<td>3</td>
<td>2/36</td>
</tr>
<tr>
<td>4</td>
<td>3/36</td>
</tr>
<tr>
<td>5</td>
<td>4/36</td>
</tr>
<tr>
<td>6</td>
<td>5/36</td>
</tr>
<tr>
<td>7</td>
<td>6/36</td>
</tr>
<tr>
<td>8</td>
<td>5/36</td>
</tr>
<tr>
<td>9</td>
<td>4/36</td>
</tr>
<tr>
<td>10</td>
<td>3/36</td>
</tr>
<tr>
<td>11</td>
<td>2/36</td>
</tr>
<tr>
<td>12</td>
<td>1/36</td>
</tr>
</tbody>
</table>

For a four-sided die and an eight-sided die there are 32 possible pairs. The probability of each possible sum is given in the following table

<table>
<thead>
<tr>
<th>Sum</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/32</td>
</tr>
<tr>
<td>3</td>
<td>2/32</td>
</tr>
<tr>
<td>4</td>
<td>3/32</td>
</tr>
<tr>
<td>5</td>
<td>4/32</td>
</tr>
<tr>
<td>6</td>
<td>4/32</td>
</tr>
<tr>
<td>7</td>
<td>4/32</td>
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<tr>
<td>8</td>
<td>4/32</td>
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<tr>
<td>9</td>
<td>4/32</td>
</tr>
<tr>
<td>10</td>
<td>3/32</td>
</tr>
<tr>
<td>11</td>
<td>2/32</td>
</tr>
<tr>
<td>12</td>
<td>1/32</td>
</tr>
</tbody>
</table>

We could make a \( 12 \times 12 \) table of probabilities, but it is not necessary. Since the sums on the two pairs of dice are independent the probabilities multiply, e.g.

\[
P(\text{the pair of 6-sided dice sum to 5 and the 8 and 4-sided dice sum to 10}) = \frac{4}{36} \cdot \frac{3}{32} = \frac{12}{1152}
\]

The probability the pair of 6-sided dice wins is

\[
P(\text{pair of 6-sided dice sum to 3})P(4 \text{ and 8-sided dice sum to 2}) + P(\text{pair of 6-sided dice sum to 4})P(4 \text{ and 8-sided dice sum to 2 or 3}) + P(\text{pair of 6-sided dice sum to 5})P(4 \text{ and 8-sided dice sum to 2, 3 or 4})
\]

\[
+ P(\text{pair of 6-sided dice sum to 6})P(4 \text{ and 8-sided dice sum to 2-5}) + P(\text{pair of 6-sided dice sum to 7})P(4 \text{ and 8-sided dice sum to 2-6})
\]

\[
+ P(\text{pair of 6-sided dice sum to 8})P(4 \text{ and 8-sided dice sum to 2-7}) + P(\text{pair of 6-sided dice sum to 9})P(4 \text{ and 8-sided dice sum to 2-8})
\]

\[
+ P(\text{pair of 6-sided dice sum to 10})P(4 \text{ and 8-sided dice sum to 2-9}) + P(\text{pair of 6-sided dice sum to 11})P(4 \text{ and 8-sided dice sum to 2-10})
\]

\[
+ P(\text{pair of 6-sided dice sum to 12})P(4 \text{ and 8-sided dice sum to 2-11})
\]

Using the tables it is easy to compute that this is

\[
\frac{2}{36} \cdot \frac{1}{36} + \frac{3}{36} \cdot \frac{3}{32} + \frac{4}{36} \cdot \frac{6}{32} + \frac{5}{36} \cdot \frac{10}{32} + \frac{6}{36} \cdot \frac{14}{32} + \frac{5}{36} \cdot \frac{18}{32} + \frac{4}{36} \cdot \frac{22}{32} + \frac{3}{36} \cdot \frac{26}{32} + \frac{2}{36} \cdot \frac{29}{32} + \frac{1}{36} \cdot \frac{31}{32} = \frac{514}{1152}
\]

(c) Here is the code I wrote to do the simulation:

```
ntrials = 10000
# Sample ntrials times from each of the dice
y20 = sample(20, ntrials, replace=True)
y4 = sample(4, ntrials, replace=True)
y6 = sample(6, ntrials, replace=True)
```
y8 = sample(8, ntrials, replace=TRUE)
y12 = sample(12, ntrials, replace=TRUE)
x = y4 + y6 + y8 + y12  # Sum the values of the 4, 6, 8, and 12, sided dice
prob = sum(y20 < x)/ntrials
print(prob)

Running this a few times, I got the following estimates for the probability of the four dice winning:

0.7734, 0.7648, 0.7709.

This was not part of the problem, but as a special treat we show that the exact calculation required to compute the probability is something that R can do as well. You need to list all $4 \cdot 6 \cdot 8 \cdot 12$ possible outcomes for the four small dice; and for each outcome, add the number of smaller rolls of the twenty-sided die. Here is code to do this:

```r
ways=0
for (a in 1:4) {
  for (b in 1:6) {
    for (c in 1:8) {
      for (d in 1:12) {
        ways = ways + (a+b+c+d-1)*(a+b+c+d <= 20) + 20*(a+b+c+d > 20)
        # This line adds to ways the number of rolls of the 20-sided die
        # that are smaller than the sum a+b+c+d.
        # That’s a+b+c+d-1 if the sum is at most 20; or 20 otherwise.
        # Note: R interprets an equality or inequality between two
        # numbers as TRUE or FALSE, represented as 1 or 0.
      }
    }
  }
}
p = ways/(20*12*8*6*4)

This gives an exact value for the probability of .77528, in good agreement with the simulations.

**Problem 3.** (20 pts.)  (a) The sample space $\Omega$ is the set of all sequences of $n$ birthdays. That is, all sequences

$$\omega = (b_1, b_2, b_3, \ldots, b_n),$$

where each entry is a number between 1 and 365.

There are $365^n$ sequences of $n$ birthdays. Since they are all equally likely, $P(\omega) = \frac{1}{365^n}$

for every sequence $\omega$.

(b) Suppose my birthday is on day $b$. Then “an outcome $\omega$ is in $A$” is equivalent to “$b$ is in the sequence for $\omega$”, i.e. $b = b_k$ for some index $k$ between 1 and $n$. More
symbolically,

an outcome \( \omega \) is in \( A \) if and only if \( b_k = b \) for some index \( k \) in \( 1, \ldots, n \).

It’s easier to calculate \( P(A^c) \). There are \( 364^n \) outcomes in \( A^c \) since there are 364 choices for birthdays that are not yours. So,

\[
P(A) = 1 - P(A^c) = 1 - \frac{364^n}{365^n}.
\]

(c) An outcome \( \omega \) is in \( B \) if “two of the entries in \( \omega \) are the same”. That is, an outcome \( \omega \) is in \( B \) if and only if \( b_j = b_k \) for two (different) indices \( j, k \) in \( 1, \ldots, n \).

It’s easier to calculate \( P(B^c) \), the probability that all \( n \) birthdays are distinct. Then there are 365 choices for the first birthday, 364 for the second birthday, etc. So

\[
P(B) = 1 - P(B^c) = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n} = 1 - \frac{365!}{(365 - n)! \cdot 365^n}.
\]

(d) No matter how large \( n \) is, there is always some probability that no one will match my birthday. So \( P(A) \) is always strictly less than 1. (Of course, it will get asymptotically close to 1 as \( n \) increases.)

For \( P(B) \), as soon as \( n \geq 366 \) there has to be at least one pair of people with the same birthday. So, for \( n \geq 366 \) we have \( P(B) = 1 \).