Exam 1 Practice Questions I – solutions, 18.05, Spring 2017

Note: This is a set of practice problems for exam 1. The actual exam will be much shorter.

1. Sort the letters: A BB II L O P R T Y. There are 11 letters in all. We build arrangements by starting with 11 ‘slots’ and placing the letters in these slots, e.g

\[
A \ B \ I \ B \ I \ L \ O \ P \ R \ T \ Y
\]

Create an arrangement in stages and count the number of possibilities at each stage:

Stage 1: Choose one of the 11 slots to put the A: \(\binom{11}{1}\)

Stage 2: Choose two of the remaining 10 slots to put the B’s: \(\binom{10}{2}\)

Stage 3: Choose two of the remaining 8 slots to put the B’s: \(\binom{8}{2}\)

Stage 4: Choose one of the remaining 6 slots to put the L: \(\binom{6}{1}\)

Stage 5: Choose one of the remaining 5 slots to put the O: \(\binom{5}{1}\)

Stage 6: Choose one of the remaining 4 slots to put the P: \(\binom{4}{1}\)

Stage 7: Choose one of the remaining 3 slots to put the R: \(\binom{3}{1}\)

Stage 8: Choose one of the remaining 2 slots to put the T: \(\binom{2}{1}\)

Stage 9: Use the last slot for the Y: \(\binom{1}{1}\)

Number of arrangements:

\[
\binom{11}{1} \cdot \binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{1} \cdot \binom{4}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} = 11 \cdot \frac{10 \cdot 9}{2} \cdot \frac{8 \cdot 7}{2} \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 9979200
\]

Note: choosing 11 out of 1 is so simple we could have immediately written 11 instead of belaboring the issue by writing \(\binom{11}{1}\). We wrote it this way to show one systematic way to think about problems like this.

2. Build the pairings in stages and count the ways to build each stage:

Stage 1: Choose the 4 men: \(\binom{6}{4}\).

Stage 2: Choose the 4 women: \(\binom{7}{4}\)
We need to be careful because we don’t want to build the same 4 couples in multiple ways. Line up the 4 men $M_1, M_2, M_3, M_4$
Stage 3: Choose a partner from the 4 women for $M_1$: 4.
Stage 4: Choose a partner from the remaining 3 women for $M_2$: 3
Stage 5: Choose a partner from the remaining 2 women for $M_3$: 2
Stage 6: Pair the last women with $M_4$: 1
Number of possible pairings: \[
\binom{6}{4} \binom{7}{4} 4!
\]
Note: we could have done stages 3-6 in on go as: Stages 3-6: Arrange the 4 women opposite the 4 men: $4!$ ways.

3. We are given $P(A^c \cap B^c) = 2/3$ and asked to find $P(A \cup B)$.

\[
A^c \cap B^c = (A \cup B)^c \Rightarrow P(A \cup B) = 1 - P(A^c \cap B^c) = \frac{1}{3}.
\]

4. $D$ is the disjoint union of $D \cap C$ and $D \cap C^c$.

So, $P(D \cap C) + P(D \cap C^c) = P(D)$

\[
\Rightarrow P(D \cap C^c) = P(D) - P(D \cap C) = 0.45 - 0.1 = 0.35.
\]

(We never use $P(C) = 0.25$.)

5. The following tree shows the setting

\[
\begin{array}{c}
\text{Know} \quad 1 - p \\
\text{Correct} \quad 0 \quad 1/c \quad \text{Wrong} \quad 1 - 1/c \quad \text{Correct} \quad \text{Wrong}
\end{array}
\]

Let $C$ be the event that you answer the question correctly. Let $K$ be the event that you actually know the answer. The left circled node shows $P(K \cap C) = p$. Both circled nodes together show $P(C) = p + (1 - p)/c$. So,

\[
P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{p}{p + (1 - p)/c}
\]

Or we could use the algebraic form of Bayes theorem and the law of total probability: Let $G$ stand for the event that you’re guessing. Then we have,

$P(C|K) = 1, P(K) = p, P(C) = P(C|K)P(K) + P(C|G)P(G) = p + (1 - p)/c$. So,

\[
P(K|C) = \frac{P(C|K)P(K)}{P(C)} = \frac{p}{p + (1 - p)/c}
\]

6. Sample space =

\[
\Omega = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 6)\} = \{(i, j) \mid i, j = 1, 2, 3, 4, 5, 6\}.
\]
(Each outcome is equally likely, with probability \(1/36\).)

\[ A = \{(1, 2), (2, 1)\} \]
\[ B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} \]
\[ C = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1)\} \]

(a) \[ P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{2/36}{11/36} = \frac{2}{11} \]

(b) \[ P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{2/36}{11/36} = \frac{2}{11} \]

(c) \[ P(A) = 2/36 \neq P(A|C), \text{ so they are not independent. Similarly, } P(B) = 6/36 \neq P(B|C), \text{ so they are not independent.} \]

7. We show the probabilities in a tree:

For a given problem let \( C \) be the event the student gets the problem correct and \( K \) the event the student knows the answer.

The question asks for \( P(K|C) \).

We’ll compute this using Bayes’ rule:

\[
P(K|C) = \frac{P(C|K) P(K)}{P(C)} = \frac{1 \cdot 1/2}{1/2 + 1/12 + 1/16} = \frac{24}{31} \approx 0.774 = 77.4\%
\]

8. We have \( P(A \cup B) = 1 - 0.42 = 0.58 \) and we know because of the inclusion-exclusion principle that

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B).
\]

Thus,

\[
P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.58 = 0.12 = (0.4)(0.3) = P(A)P(B)
\]

So \( A \) and \( B \) are independent.

9. We will make use of the formula \( \text{Var}(Y) = E(Y^2) - E(Y)^2 \). First we compute

\[
E[X] = \int_0^1 x \cdot 2x \, dx = \frac{2}{3}
\]
\[
E[X^2] = \int_0^1 x^2 \cdot 2x \, dx = \frac{1}{2}
\]
Thus,
\[ \text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}, \]

and
\[ \text{Var}(X^2) = E[X^4] - (E[X^2])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \]

10. Make a table

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>prob:</td>
<td>1-p</td>
<td>p</td>
</tr>
<tr>
<td>( X^2 )</td>
<td>0</td>
<td>1.</td>
</tr>
</tbody>
</table>

From the table, \( E(X) = 0 \cdot (1 - p) + 1 \cdot p = p \).
Since \( X \) and \( X^2 \) have the same table \( E(X^2) = E(X) = p \).
Therefore, \( \text{Var}(X) = p - p^2 = p(1-p) \).

11. Let \( X \) be the number of people who get their own hat.
Following the hint: let \( X_j \) represent whether person \( j \) gets their own hat. That is, \( X_j = 1 \) if person \( j \) gets their hat and 0 if not.
We have, \( X = \sum_{j=1}^{100} X_j \), so \( E(X) = \sum_{j=1}^{100} E(X_j) \).
Since person \( j \) is equally likely to get any hat, we have \( P(X_j = 1) = 1/100 \). Thus, \( X_j \sim \text{Bernoulli}(1/100) \Rightarrow E(X_j) = 1/100 \Rightarrow E(X) = 1 \).

12. For \( y = 0, 2, 4, \ldots, 2n \),
\[ P(Y = y) = P(X = \frac{y}{2}) = \binom{n}{y/2} \left( \frac{1}{2} \right)^n. \]

13. The CDF for \( R \) is
\[ F_R(r) = P(R \leq r) = \int_0^r 2e^{-2u} \, du = -e^{-2u}\big|_0^r = 1 - e^{-2r}. \]
Next, we find the CDF of \( T \). \( T \) takes values in \((0, \infty)\).
For \( 0 < t \),
\[ F_T(t) = P(T \leq t) = P(1/R < t) = P(1/t > R) = 1 - F_R(1/t) = e^{-2/t}. \]
We differentiate to get \( f_T(t) = \frac{d}{dt} \left( e^{-2/t} \right) = \frac{2}{t^2} e^{-2/t} \).

14. The jumps in the distribution function are at 0, 2, 4. The value of \( p(a) \) at a jump is the height of the jump:
15. We compute

\[ P(X \geq 5) = 1 - P(X < 5) = 1 - \int_{0}^{5} \lambda e^{-\lambda x} dx = 1 - (1 - e^{-5\lambda}) = e^{-5\lambda}. \]

(b) We want \( P(X \geq 15 | X \geq 10) \). First observe that \( P(X \geq 15, X \geq 10) = P(X \geq 15) \). From similar computations in (a), we know

\[ P(X \geq 15) = e^{-15\lambda} \quad P(X \geq 10) = e^{-10\lambda}. \]

From the definition of conditional probability,

\[ P(X \geq 15 | X \geq 10) = \frac{P(X \geq 15, X \geq 10)}{P(X \geq 10)} = \frac{P(X \geq 15)}{P(X \geq 10)} = e^{-5\lambda} \]

Note: This is an illustration of the memorylessness property of the exponential distribution.

16. Transforming Normal Distributions
(a) Note, \( Y \) follows what is called a log-normal distribution.

\[ F_Y(a) = P(Y \leq a) = P(e^Z \leq a) = P(Z \leq \ln(a)) = \Phi(\ln(a)). \]

Differentiating using the chain rule:

\[ f_y(a) = \frac{d}{da} F_Y(a) = \frac{d}{da} \Phi(\ln(a)) = \frac{1}{a} \phi(\ln(a)) = \frac{1}{\sqrt{2\pi} a} e^{-(\ln(a))^2/2}. \]

(b) (i) The 0.33 quantile for \( Z \) is the value \( q_{0.33} \) such that \( P(Z \leq q_{0.33}) = 0.33 \). That is, we want

\[ \Phi(q_{0.33}) = 0.33 \iff q_{0.33} = \Phi^{-1}(0.33). \]

(ii) We want to find \( q_{0.9} \) where

\[ F_Y(q_{0.9}) = 0.9 \iff \Phi(\ln(q_{0.9})) = 0.9 \iff q_{0.9} = e^{\Phi^{-1}(0.9)}. \]

(iii) As in (ii) \( q_{0.5} = e^{\Phi^{-1}(0.5)} = e^0 = 1 \).

17. (a) Total probability must be 1, so

\[ 1 = \int_{0}^{3} \int_{0}^{3} f(x, y) dy dx = \int_{0}^{3} \int_{0}^{3} c(x^2 y + x^2 y^2) dy dx = c \cdot \frac{243}{2}, \]

(Here we skipped showing the arithmetic of the integration) Therefore, \( c = \frac{2}{243} \).
(b) 
\[ P(1 \leq X \leq 2, 0 \leq Y \leq 1) = \int_{0}^{1} \int_{1}^{2} f(x, y) \, dy \, dx \]
\[ = \int_{0}^{1} \int_{1}^{2} c(x^2 y + x^2 y^2) \, dy \, dx \]
\[ = c \cdot \frac{35}{18} \]
\[ = \frac{70}{4374} \approx 0.016 \]

(c) For \( 0 \leq a \leq 1 \) and \( 0 \leq b \leq 1 \), we have
\[ F(a, b) = \int_{0}^{a} \int_{0}^{b} f(x, y) \, dy \, dx = c \left( \frac{a^3 b^2}{6} + \frac{a^3 b^3}{9} \right) \]

(d) Since \( y = 3 \) is the maximum value for \( Y \), we have
\[ F_X(a) = F(a, 3) = c \left( \frac{9a^3}{6} + 3a^3 \right) = \frac{9}{2} c a^3 = \frac{a^3}{27} \]

(e) For \( 0 \leq x \leq 3 \), we have, by integrating over the entire range for \( y \),
\[ f_X(x) = \int_{0}^{3} f(x, y) \, dy = cx^2 \left( \frac{3^2}{2} + \frac{3^3}{3} \right) = c \frac{27}{2} x^2 = \frac{1}{9} x^2. \]

This is consistent with (c) because \( \frac{d}{dx}(x^3/27) = x^2/9 \).

(f) Since \( f(x, y) \) separates into a product as a function of \( x \) times a function of \( y \) we know \( X \) and \( Y \) are independent.

18. **(Central Limit Theorem)** Let \( T = X_1 + X_2 + \ldots + X_{81} \). The central limit theorem says that
\[ T \approx N(81 \times 5, 81 \times 4) = N(405, 18^2) \]

Standardizing we have
\[ P(T > 369) = P \left( \frac{T - 405}{18} > \frac{369 - 405}{18} \right) \approx P(Z > -2) \approx 0.975 \]

The value of 0.975 comes from the rule-of-thumb that \( P(|Z| < 2) \approx 0.95 \). A more exact value (using R) is \( P(Z > -2) \approx 0.9772 \).