Class 8 Review Problems
18.05, Spring 2018

1 Counting and Probability

1. **(a)** How many ways can you arrange the letters in the word STATISTICS? (e.g. SSSTTTIIAC counts as one arrangement.)

(b) If all arrangements are equally likely, what is the probability the two 'i's are next to each other?

2. (Taken from the book by Dekking *et al.* problem 4.9) The space shuttle has 6 O-rings (these were involved in the *Challenger* disaster). When launched at 81° F, each O-ring has a probability of failure of 0.0137 (independent of whether other O-rings fail).
(a) What is the probability that during 23 launches no O-ring will fail, but that at least one O-ring will fail during the 24th launch of a space shuttle?

(b) What is the probability that no O-ring fails during 24 launches?

2 Conditional Probability and Bayes' Theorem

3. Corrupted by their power, the judges running the popular game show *America's Next Top Mathematician* have been taking bribes from many of the contestants. Each episode, a given contestant is either allowed to stay on the show or is kicked off.

If the contestant has been bribing the judges she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3.

Over two rounds, suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds, i.e., if a contestant bribes them in the first round, she bribes them in the second round too (and vice versa).

(a) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

(b) If you pick a random contestant, what is the probability that she is allowed to stay during both of the first two episodes?

(c) If you pick random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

3 Independence

4. You roll a twenty-sided die. Determine whether the following pairs of events are independent.

(a) 'You roll an even number' and 'You roll a number less than or equal to 10'.

(b) 'You roll an even number' and 'You roll a prime number'.

4 Expectation and Variance

5. The random variable X takes values -1, 0, 1 with probabilities 1/8, 2/8, 5/8 respectively.

(a) Compute $E(X)$.

(b) Give the pmf of $Y = X^2$ and use it to compute $E(Y)$.

(c) Instead, compute $E(X^2)$ directly from an extended table.

(d) Compute $\text{Var}(X)$.

6. Compute the expectation and variance of a Bernoulli(p) random variable.

7. Suppose 100 people all toss a hat into a box and then proceed to randomly pick out a hat. What is the expected number of people to get their own hat back.

Hint: express the number of people who get their own hat as a sum of random variables whose expected value is easy to compute.

5 Probability Mass Functions, Probability Density Functions and Cumulative Distribution Functions

8. **(a)** Suppose that X has probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Compute the cdf, $F_X(x)$.

(b) If $Y = X^2$, compute the pdf and cdf of Y.

9. Suppose you roll a fair 6-sided die 100 times (independently), and you get \$3 every time you roll a 6. Let X_1 be the number of dollars you win on rolls 1 through 25.

Let X_2 be the number of dollars you win on rolls 26 through 50.

Let X_3 be the number of dollars you win on rolls 51 through 75.

Let X_4 be the number of dollars you win on rolls 76 throught 100.

Let $X = X_1 + X_2 + X_3 + X_4$ be the total number of dollars you win over all 100 rolls.

(a) What is the probability mass function of X?

(b) What is the expectation and variance of X?

(c) Let $Y = 4X_1$. (So instead of rolling 100 times, you just roll 25 times and multiply your winnings by 4.) (i) What are the expectation and variance of Y?

(ii) How do the expectation and variance of Y compare to those of X? (I.e., are they bigger, smaller, or equal?) Explain (briefly) why this makes sense.

6 Joint Probability, Covariance, Correlation

10. (**Arithmetic Puzzle**) The joint and marginal pmf's of X and Y are partly given in the following table.

$X \backslash Y$	1	2	3	
1	1/6	0	...	1/3
2	...	1/4	...	1/3
3	...	...	1/4	...
	1/6	1/3	...	1

(a) Complete the table.

(b) Are X and Y independent?

11. Covariance and Independence
Let X be a random variable that takes values -2, -1, 0, 1, 2; each with probability 1/5. Let $Y = X^2$.

(a) Fill out the following table giving the joint frequency function for X and Y. Be sure to include the marginal probabilities.

X / Y	-2	-1	0	1	2	total
0						
1						
4						
total						

(b) Find $E(X)$ and $E(Y)$.

(c) Show X and Y are not independent.

(d) Show $\text{Cov}(X, Y) = 0$.
This is an example of uncorrelated but non-independent random variables. The reason this can happen is that correlation only measures the linear dependence between the two variables. In this case, X and Y are not at all linearly related.

12. Continuous Joint Distributions
Suppose X and Y are continuous random variables with joint density function $f(x, y) = x + y$ on the unit square $[0, 1] \times [0, 1]$.

(a) Let $F(x, y)$ be the joint CDF. Compute $F(1, 1)$. Compute $F(x, y)$.

(b) Compute the marginal densities for X and Y.

(c) Are X and Y independent?

(d) Compute $E(X)$, (Y), $E(X^2 + Y^2)$, $\text{Cov}(X, Y)$.

7 Law of Large Numbers, Central Limit Theorem

13. Suppose $X_1, \ldots, X_{100}$ are i.i.d. with mean 1/5 and variance 1/9. Use the central limit theorem to estimate $P(\sum X_i < 30)$.

14. (**More Central Limit Theorem**)
The average IQ in a population is 100 with standard deviation 15 (by definition, IQ is normalized so this is the case). What is the probability that a randomly selected group of 100 people has an average IQ above 115?