Central Limit Theorem, Joint Distributions
18.05 Spring 2018
Exam next Wednesday

- Exam 1 on Wednesday March 7, regular room and time.
- Designed for 1 hour. You will have the full 80 minutes.
- Class on Monday will be review.
- Practice materials posted.
- Learn to use the standard normal table for the exam.
- No books or calculators.
- You may have one 4 × 6 notecard with any information you like.
The bell-shaped curve

- This is standard normal distribution $N(0, 1)$:

$$
\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}
$$

- $N(0, 1)$ means that mean is $\mu = 0$, and std deviation is $\sigma = 1$.
- Normal with mean $\mu$, std deviation $\sigma$ is $N(\mu, \sigma)$:

$$
\phi_{\mu, \sigma}(z) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(z-\mu)^2/2\sigma^2}
$$
Lots of normal distributions

- $N(0, 1)$
- $N(4.5, 0.5)$
- $N(4.5, 2.25)$
- $N(6.5, 1.0)$
- $N(8.0, 0.5)$
Random variable $X$ with mean $\mu$, standard deviation $\sigma$.

**Standardization:**

$$Y = \frac{X - \mu}{\sigma}.$$  

- $Y$ has mean 0 and standard deviation 1.
- Standardizing any normal random variable produces the standard normal.
- If $X \approx \text{normal}$ then standardized $X \approx \text{stand. normal}$.
- We reserve $Z$ to mean a standard normal random variable.
Board Question: Standardization

Here are the pdfs for four (binomial) random variables $X$. Standardize them, and make bar graphs of the standardized distributions. Each bar should have area equal to the probability of that value. (Each bar has width $1/\sigma$, so each bar has height $pdf \cdot \sigma$.)

<table>
<thead>
<tr>
<th>$X$</th>
<th>$n = 0$</th>
<th>$n = 1$</th>
<th>$n = 4$</th>
<th>$n = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>1/16</td>
<td>1/512</td>
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</tbody>
</table>
Concept Question: Normal Distribution

$X$ has normal distribution, standard deviation $\sigma$.

![Normal PDF](image)

1. $P(-\sigma < X < \sigma)$ is
   (a) 0.025  (b) 0.16  (c) 0.68  (d) 0.84  (e) 0.95
Concept Question: Normal Distribution

$X$ has normal distribution, standard deviation $\sigma$.

1. $P(-\sigma < X < \sigma)$ is
   (a) 0.025    (b) 0.16    (c) 0.68    (d) 0.84    (e) 0.95

2. $P(X > 2\sigma)$
   (a) 0.025    (b) 0.16    (c) 0.68    (d) 0.84    (e) 0.95
Central Limit Theorem

**Setting:** $X_1, X_2, \ldots \text{i.i.d. with mean } \mu \text{ and standard dev. } \sigma$.

For each $n$:

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \ldots + X_n) \quad \text{average}$$

$$S_n = X_1 + X_2 + \ldots + X_n \quad \text{sum.}$$

**Conclusion:** For large $n$:

$$\bar{X}_n \approx N \left( \mu, \frac{\sigma^2}{n} \right)$$

$$S_n \approx N \left( n\mu, n\sigma^2 \right)$$

Standardized $(S_n$ or $\bar{X}_n) \approx N(0, 1)$

That is, $\frac{S_n - n\mu}{\sqrt{n\sigma}} = \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \approx N(0, 1)$. 
CLT: pictures

The standardized average of $n$ i.i.d. Bernoulli(0.5) random variables with $n = 1, 2, 12, 64$. 

![Graphs showing the distribution of standardized averages for different values of $n$.]
CLT: pictures 2

Standardized average of $n$ i.i.d. uniform random variables with $n = 1, 2, 4, 12$. 
CLT: pictures 3

The standardized average of \( n \) i.i.d. exponential random variables with \( n = 1, 2, 8, 64 \).
The non-standardized average of $n$ Bernoulli(0.5) random variables, with $n = 4, 12, 64$. Spikier.
Table Question: Sampling from the standard normal distribution

As a table, produce two random samples from (an approximate) standard normal distribution.

To make each sample, the table is allowed eight rolls of the 10-sided die.

**Note:** $\mu = 5.5$ and $\sigma^2 \approx 8$ for a single 10-sided die.

**Hint:** CLT is about averages.
Board Question: CLT

1. Carefully write the statement of the central limit theorem.

2. To head the newly formed US Dept. of Statistics, suppose that 50% of the population supports Ani, 25% supports Ruthi, and the remaining 25% is split evenly between Efrat, Elan, David and Jerry. A poll asks 400 random people who they support. What is the probability that at least 55% of those polled prefer Ani?

3. What is the probability that less than 20% of those polled prefer Ruthi?
An accountant rounds to the nearest dollar. We’ll assume the error in rounding is uniform on $[-0.5, 0.5]$. Estimate the probability that the total error in 300 entries is more than $5$. 

Not for class. Solution will be posted with the slides.
Joint Distributions

$X$ and $Y$ are jointly distributed random variables.

Discrete: Probability mass function (pmf):

$$p(x_i, y_j)$$

Continuous: probability density function (pdf):

$$f(x, y)$$

Both: cumulative distribution function (cdf):

$$F(x, y) = P(X \leq x, Y \leq y)$$
Discrete joint pmf: example 1

Roll two dice: \( X = \# \text{ on first die}, \ Y = \# \text{ on second die} \)

\( X \) takes values in 1, 2, \ldots, 6, \( Y \) takes values in 1, 2, \ldots, 6

**Joint probability table:**

\[
\begin{array}{c|cccccc}
 X \backslash Y & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
 1 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
 2 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
 3 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
 4 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
 5 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
 6 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
\end{array}
\]

pmf: \( p(i,j) = 1/36 \) for any \( i \) and \( j \) between 1 and 6.
Discrete joint pmf: example 2

Roll two dice: $X = \#$ on first die, $T = \text{total on both dice}$

<table>
<thead>
<tr>
<th>$X \setminus T$</th>
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Continuous joint distributions

- $X$ takes values in $[a, b]$, $Y$ takes values in $[c, d]$.
- $(X, Y)$ takes values in $[a, b] \times [c, d]$.
- Joint probability density function (pdf) $f(x, y)$

$f(x, y) \, dx \, dy$ is the probability of being in the small square.
Properties of the joint pmf and pdf

**Discrete case: probability mass function (pmf)**
1. $0 \leq p(x_i, y_j) \leq 1$
2. Total probability is 1:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

**Continuous case: probability density function (pdf)**
1. $0 \leq f(x, y)$
2. Total probability is 1:

$$\int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy = 1$$

Note: $f(x, y)$ can be greater than 1: it is a density, *not* a probability.
Example: discrete events
Roll two dice: $X = \#$ on first die, $Y = \#$ on second die.
Consider the event: $A = 'Y - X \geq 2'$
Describe the event $A$ and find its probability.
Example: discrete events
Roll two dice: $X = \#$ on first die, $Y = \#$ on second die.
Consider the event: $A = 'Y - X \geq 2'$
Describe the event $A$ and find its probability.

answer: We can describe $A$ as a set of $(X, Y)$ pairs:

$$A = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6), (4, 6)\}.$$ 

Or we can visualize it by shading the table:

<table>
<thead>
<tr>
<th>$X \backslash Y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
</tr>
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<td>1</td>
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</tr>
</tbody>
</table>

$P(A) = \text{sum of probabilities in shaded cells} = 10/36.$
Example: continuous events

Suppose \((X, Y)\) takes values in \([0, 1] \times [0, 1]\).

Uniform density \(f(x, y) = 1\).

Visualize the event ‘\(X > Y\)’ and find its probability.
Example: continuous events

Suppose \((X, Y)\) takes values in \([0, 1] \times [0, 1]\).

Uniform density \(f(x, y) = 1\).

Visualize the event ‘\(X > Y\)’ and find its probability.

**answer:**

The event takes up half the square. Since the density is uniform this is half the probability. That is, \(P(X > Y) = 0.5\).
Cumulative distribution function

\[ F(x, y) = P(X \leq x, Y \leq y) = \int_{a}^{x} \int_{c}^{y} f(u, v) \, du \, dv. \]

\[ f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y). \]

Properties

1. \( F(x, y) \) is non-decreasing. That is, as \( x \) or \( y \) increases \( F(x, y) \) increases or remains constant.

2. \( F(x, y) = 0 \) at the lower left of its range.
   If the lower left is \((-\infty, -\infty)\) then this means
   \[
   \lim_{(x,y) \to (-\infty, -\infty)} F(x, y) = 0.
   \]

3. \( F(x, y) = 1 \) at the upper right of its range.
Marginal pmf and pdf

Roll two dice: $X = \#$ on first die, $T = \text{total}$ on both dice.

The marginal pmf of $X$ is found by summing the rows. The marginal pmf of $T$ is found by summing the columns.

<table>
<thead>
<tr>
<th>$X \setminus T$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</table>

| $p(x_i)$        |      |      |      |      |      |      |      |      |      |      | 1/6  |


For continuous distributions the marginal pdf $f_X(x)$ is found by integrating out the $y$. Likewise for $f_Y(y)$. 
Suppose $X$ and $Y$ are random variables and
- $(X, Y)$ takes values in $[0, 1] \times [0, 1]$.
- the pdf is \( \frac{3}{2}(x^2 + y^2) \).

1. Show $f(x, y)$ is a valid pdf.
2. Visualize the event $A = \{X > 0.3 \text{ and } Y > 0.5\}$. Find its probability.
3. Find the cdf $F(x, y)$.
4. Find the marginal pdf $f_X(x)$. Use this to find $P(X < 0.5)$.
5. Use the cdf $F(x, y)$ to find the marginal cdf $F_X(x)$ and $P(X < 0.5)$.
6. See next slide
6. (New scenario) From the following table compute $F(3.5, 4)$.

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
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