Central Limit Theorem, Joint Distributions
18.05 Spring 2018
Exam next Wednesday

- Exam 1 on Wednesday March 7, regular room and time.
- Designed for 1 hour. You will have the full 80 minutes.
- Class on Monday will be review.
- Practice materials posted.
- Learn to use the standard normal table for the exam.
- No books or calculators.
- You may have one 4 × 6 notecard with any information you like.
The bell-shaped curve

This is standard normal distribution \( N(0, 1) \):

\[
\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}
\]

\( N(0, 1) \) means that mean is \( \mu = 0 \), and std deviation is \( \sigma = 1 \).

Normal with mean \( \mu \), std deviation \( \sigma \) is \( N(\mu, \sigma) \):

\[
\phi_{\mu, \sigma}(z) = \frac{1}{\sigma \sqrt{2\pi}} e^{-{(z-\mu)^2}/2\sigma^2}
\]
Lots of normal distributions

- $N(0, 1)$
- $N(4.5, 0.5)$
- $N(4.5, 2.25)$
- $N(6.5, 1.0)$
- $N(8.0, 0.5)$
Random variable $X$ with mean $\mu$, standard deviation $\sigma$.

**Standardization:**

$$Y = \frac{X - \mu}{\sigma}.$$

- $Y$ has mean 0 and standard deviation 1.
- Standardizing any normal random variable produces the standard normal.
- If $X \approx \text{normal}$ then standardized $X \approx \text{stand. normal}$.
- We reserve $Z$ to mean a standard normal random variable.
Board Question: Standardization

Here are the pdfs for four (binomial) random variables $X$. Standardize them, and make bar graphs of the standardized distributions. Each bar should have area equal to the probability of that value. (Each bar has width $1/\sigma$, so each bar has height $\text{pdf} \cdot \sigma$.)

<table>
<thead>
<tr>
<th>$X$</th>
<th>$n = 0$</th>
<th>$n = 1$</th>
<th>$n = 4$</th>
<th>$n = 9$</th>
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<tbody>
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<td>1/16</td>
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</table>
Concept Question: Normal Distribution

$X$ has normal distribution, standard deviation $\sigma$. 

1. $P(-\sigma < X < \sigma)$ is 
   (a) 0.025  (b) 0.16  (c) 0.68  (d) 0.84  (e) 0.95

2. $P(X > 2\sigma)$
   (a) 0.025  (b) 0.16  (c) 0.68  (d) 0.84  (e) 0.95

answer: 1c, 2a
Central Limit Theorem

**Setting:** \( X_1, X_2, \ldots \) i.i.d. with mean \( \mu \) and standard dev. \( \sigma \).

For each \( n \):

\[
\overline{X}_n = \frac{1}{n}(X_1 + X_2 + \ldots + X_n) \quad \text{average}
\]

\[
S_n = X_1 + X_2 + \ldots + X_n \quad \text{sum.}
\]

**Conclusion:** For large \( n \):

\[
\overline{X}_n \approx N \left( \mu, \frac{\sigma^2}{n} \right)
\]

\[
S_n \approx N \left( n\mu, n\sigma^2 \right)
\]

Standardized \((S_n \text{ or } \overline{X}_n) \approx N(0, 1)\)

That is, \[
\frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \approx N(0, 1).
\]
The standardized average of $n$ i.i.d. Bernoulli(0.5) random variables with $n = 1, 2, 12, 64$. 

![Graphs showing the distribution for different $n$ values.](image-url)
Standardized average of $n$ i.i.d. uniform random variables with $n = 1, 2, 4, 12$. 
The standardized average of $n$ i.i.d. exponential random variables with $n = 1, 2, 8, 64$. 
CLT: pictures

The **non-standardized** average of $n$ Bernoulli(0.5) random variables, with $n = 4, 12, 64$. **Spikier.**
Table Question: Sampling from the standard normal distribution

As a table, produce two random samples from (an approximate) standard normal distribution.

To make each sample, the table is allowed eight rolls of the 10-sided die.

Note: $\mu = 5.5$ and $\sigma^2 \approx 8$ for a single 10-sided die.

Hint: CLT is about averages.

answer: The average of 9 rolls is a sample from the average of 9 independent random variables. The CLT says this average is approximately normal with $\mu = 5.5$ and $\sigma = 8.25/\sqrt{9} = 2.75$

If $\bar{x}$ is the average of 9 rolls then standardizing we get

$$z = \frac{\bar{x} - 5.5}{2.75}$$

is (approximately) a sample from $N(0, 1)$. 
1. Carefully write the statement of the central limit theorem.

2. To head the newly formed US Dept. of Statistics, suppose that 50% of the population supports Ani, 25% supports Ruthi, and the remaining 25% is split evenly between Efrat, Elan, David and Jerry.

A poll asks 400 random people who they support. What is the probability that at least 55% of those polled prefer Ani?

3. What is the probability that less than 20% of those polled prefer Ruthi?

**answer:** On next slide.
**Solution**

**answer:** 2. Let $A$ be the fraction polled who support Ani. So $A$ is the average of 400 Bernoulli(0.5) random variables. That is, let $X_i = 1$ if the $i$th person polled prefers Ani and 0 if not, so $A = \text{average of the } X_i$. The question asks for the probability $A > 0.55$.

Each $X_i$ has $\mu = 0.5$ and $\sigma^2 = 0.25$. So, $E(A) = 0.5$ and $\sigma_A^2 = 0.25/400$ or $\sigma_A = 1/40 = 0.025$.

Because $A$ is the average of 400 Bernoulli(0.5) variables the CLT says it is approximately normal and standardizing gives

$$\frac{A - 0.5}{0.025} \approx Z$$

So

$$P(A > 0.55) \approx P(Z > 2) \approx 0.025$$

*Continued on next slide*
3. Let $R$ be the fraction polled who support Ruthi. The question asks for the probability the $R < 0.2$. Similar to problem 2, $R$ is the average of 400 Bernoulli(0.25) random variables. So

$$E(R) = 0.25 \quad \text{and} \quad \sigma^2_R = (0.25)(0.75)/400 = \Rightarrow \sigma_R = \sqrt{3}/80.$$

So $\frac{R - 0.25}{\sqrt{3}/80} \approx Z$. So,

$$P(R < 0.2) \approx P(Z < -4/\sqrt{3}) \approx 0.0105$$
An accountant rounds to the nearest dollar. We’ll assume the error in rounding is uniform on \([-0.5, 0.5]\). Estimate the probability that the total error in 300 entries is more than $5.

**answer:** Let \(X_j\) be the error in the \(j\)th entry, so, \(X_j \sim U(-0.5, 0.5)\).

We have \(E(X_j) = 0\) and \(\text{Var}(X_j) = 1/12\).

The total error \(S = X_1 + \ldots + X_{300}\) has \(E(S) = 0\), \(\text{Var}(S) = 300/12 = 25\), and \(\sigma_S = 5\).

Standardizing we get, by the CLT, \(S/5\) is approximately standard normal. That is, \(S/5 \approx Z\).

So \(P(S < -5 \text{ or } S > 5) \approx P(Z < -1 \text{ or } Z > 1) \approx 0.32\).
Joint Distributions

$X$ and $Y$ are jointly distributed random variables.

Discrete: Probability mass function (pmf):

$$p(x_i, y_j)$$

Continuous: probability density function (pdf):

$$f(x, y)$$

Both: cumulative distribution function (cdf):

$$F(x, y) = P(X \leq x, Y \leq y)$$
Discrete joint pmf: example 1

Roll two dice: \( X = \# \) on first die, \( Y = \# \) on second die

\( X \) takes values in 1, 2, \ldots, 6, \( Y \) takes values in 1, 2, \ldots, 6

Joint probability table:

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<tr>
<th>( X )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
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pmf: \( p(i, j) = 1/36 \) for any \( i \) and \( j \) between 1 and 6.
Discrete joint pmf: example 2

Roll two dice: $X = \#$ on first die, $T = \text{total on both dice}$

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<tr>
<th>$X$</th>
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</table>
Continuous joint distributions

- $X$ takes values in $[a, b]$, $Y$ takes values in $[c, d]$
- $(X, Y)$ takes values in $[a, b] \times [c, d]$.
- Joint probability density function (pdf) $f(x, y)$

$f(x, y) \, dx \, dy$ is the probability of being in the small square.

Prob. $= f(x, y) \, dx \, dy$
Properties of the joint pmf and pdf

**Discrete case: probability mass function (pmf)**
1. $0 \leq p(x_i, y_j) \leq 1$
2. Total probability is 1:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

**Continuous case: probability density function (pdf)**
1. $0 \leq f(x, y)$
2. Total probability is 1:

$$\int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = 1$$

Note: $f(x, y)$ can be greater than 1: it is a density, *not* a probability.
Example: discrete events
Roll two dice: $X = \#$ on first die, $Y = \#$ on second die.
Consider the event: $A = 'Y - X \geq 2'$
Describe the event $A$ and find its probability.
answer: We can describe $A$ as a set of $(X, Y)$ pairs:

$$A = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6), (4, 6)\}.$$

Or we can visualize it by shading the table:

$$
\begin{array}{cccccc}
X \backslash Y & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\
2 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\
3 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\
4 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\
5 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\
6 & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\
\end{array}
$$

$P(A) = \text{sum of probabilities in shaded cells} = \frac{10}{36}$. 
Example: continuous events

Suppose \((X, Y)\) takes values in \([0, 1] \times [0, 1]\).

Uniform density \(f(x, y) = 1\).

Visualize the event \('X > Y'\) and find its probability.

\textbf{answer:}

The event takes up half the square. Since the density is uniform this is half the probability. That is, \(P(X > Y) = 0.5\).
Cumulative distribution function

\[ F(x, y) = P(X \leq x, Y \leq y) = \int_c^y \int_a^x f(u, v) \, du \, dv. \]

\[ f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y). \]

Properties

1. \( F(x, y) \) is non-decreasing. That is, as \( x \) or \( y \) increases \( F(x, y) \) increases or remains constant.

2. \( F(x, y) = 0 \) at the lower left of its range.
   If the lower left is \((-\infty, -\infty)\) then this means
   \[ \lim_{(x, y) \to (-\infty, -\infty)} F(x, y) = 0. \]

3. \( F(x, y) = 1 \) at the upper right of its range.
**Marginal pmf and pdf**

Roll two dice: $X = \#$ on first die, $T = \text{total on both dice}$.

The marginal pmf of $X$ is found by summing the rows. The marginal pmf of $T$ is found by summing the columns.

<table>
<thead>
<tr>
<th>$X \backslash T$</th>
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<th>3</th>
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</table>

$p(x_i)$


For continuous distributions the marginal pdf $f_X(x)$ is found by integrating out the $y$. Likewise for $f_Y(y)$.
Board question

Suppose $X$ and $Y$ are random variables and:

- $(X, Y)$ takes values in $[0, 1] \times [0, 1]$.
- the pdf is $\frac{3}{2}(x^2 + y^2)$.

1. Show $f(x, y)$ is a valid pdf.
2. Visualize the event $A = 'X > 0.3 \text{ and } Y > 0.5'$. Find its probability.
3. Find the cdf $F(x, y)$.
4. Find the marginal pdf $f_X(x)$. Use this to find $P(X < 0.5)$.
5. Use the cdf $F(x, y)$ to find the marginal cdf $F_X(x)$ and $P(X < 0.5)$.
6. See next slide
Board question continued

6. (New scenario) From the following table compute \( F(3.5, 4) \).

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<thead>
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<th></th>
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</tbody>
</table>

**answer:** See next slide
Solution

**answer:** 1. Validity: Clearly $f(x, y)$ is positive. Next we must show that total probability $= 1$:

$$
\int_0^1 \int_0^1 \frac{3}{2} (x^2 + y^2) \, dx \, dy = \int_0^1 \left[ \frac{1}{2} x^3 + \frac{3}{2} xy^2 \right]_0^1 \, dy = \int_0^1 \frac{1}{2} + \frac{3}{2} y^2 \, dy = 1.
$$

2. Here’s the visualization

![Diagram](image)

The pdf is not constant so we must compute an integral

$$
P(A) = \int_{.3}^1 \int_{.5}^1 \frac{3}{2} (x^2 + y^2) \, dy \, dx = \int_{.3}^1 \left[ \frac{3}{2} x^2 y + \frac{1}{2} y^3 \right]_{.5}^1 \, dx
$$

(continued)
Solutions 2, 3, 4, 5

2. (continued) \[ = \int_{0.3}^{1} \frac{3x^2}{4} + \frac{7}{16} \, dx = 0.5495 \]

3. \[ F(x, y) = \int_{0}^{y} \int_{0}^{x} \frac{3}{2} (u^2 + v^2) \, du \, dv = \frac{x^3 y}{2} + \frac{xy^3}{2}. \]

4. \[ f_X(x) = \int_{0}^{1} \frac{3}{2} (x^2 + y^2) \, dy = \left[ \frac{3}{2} x^2 y + \frac{y^3}{2} \right]_{0}^{1} = \frac{3}{2} x^2 + \frac{1}{2} \]

\[ P(X < .5) = \int_{0}^{.5} f_X(x) \, dx = \int_{0}^{.5} \frac{3}{2} x^2 + \frac{1}{2} \, dx = \left[ \frac{1}{2} x^3 + \frac{1}{2} x \right]_{0}^{.5} = \frac{5}{16}. \]

5. To find the marginal cdf \( F_X(x) \) we simply take \( y \) to be the top of the \( y \)-range and evaluate \( F \): \[ F_X(x) = F(x, 1) = \frac{1}{2} (x^3 + x). \]

Therefore \[ P(X < .5) = F(.5) = \frac{1}{2} \left( \frac{1}{8} + \frac{1}{2} \right) = \frac{5}{16}. \]

6. On next slide
6. $F(3.5, 4) = P(X \leq 3.5, Y \leq 4)$.

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<tr>
<th>$X \backslash Y$</th>
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Add the probability in the shaded squares: $F(3.5, 4) = \frac{12}{36} = \frac{1}{3}$. 