Conditional Probability, Independence, Bayes’ Theorem
18.05 Spring 2018
Slides are Posted

Don’t forget that after class we post the slides including solutions to all the questions.
Conditional Probability

‘the probability of $A$ given $B$’.

\[
P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0.
\]

Conditional probability: Abstractly and for coin example
Toss a coin 4 times. Let
$A = \text{‘at least three heads’}$
$B = \text{‘first toss is tails’}$.

1. What is $P(A|B)$?
   (a) $1/16$  (b) $1/8$  (c) $1/4$  (d) $1/5$

2. What is $P(B|A)$?
   (a) $1/16$  (b) $1/8$  (c) $1/4$  (d) $1/5$
“Steve is very shy and withdrawn, invariably helpful, but with little interest in people, or in the world of reality. A meek and tidy soul, he has a need for order and structure and a passion for detail.”

What is the probability that Steve is a librarian?
What is the probability that Steve is a farmer?

*From Judgment under uncertainty: heuristics and biases by Tversky and Kahneman.
Multiplication Rule, Law of Total Probability

Multiplication rule: \( P(A \cap B) = P(A|B) \cdot P(B) \).

Law of total probability: If \( B_1, B_2, B_3 \) partition \( \Omega \) then

\[
P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)
\]

\[
= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)
\]
Example. : Game: 5 red and 2 green balls in an urn. A random ball is selected and replaced by a ball of the other color; then a second ball is drawn.

1. What is the probability the second ball is red?
2. What is the probability the first ball was red given the second ball was red?

First draw

Second draw
1. The probability $x$ represents

(a) $P(A_1)$
(b) $P(A_1 \mid B_2)$
(c) $P(B_2 \mid A_1)$
(d) $P(C_1 \mid B_2 \cap A_1)$. 
2. The probability \( y \) represents

(a) \( P(B_2) \)
(b) \( P(A_1 \mid B_2) \)
(c) \( P(B_2 \mid A_1) \)
(d) \( P(C_1 \mid B_2 \cap A_1) \).
3. The probability $z$ represents

(a) $P(C_1)$
(b) $P(B_2 \mid C_1)$
(c) $P(C_1 \mid B_2)$
(d) $P(C_1 \mid B_2 \cap A_1)$. 
4. The circled node represents the event

(a) \( C_1 \)
(b) \( B_2 \cap C_1 \)
(c) \( A_1 \cap B_2 \cap C_1 \)
(d) \( C_1 \mid B_2 \cap A_1 \).
Let’s Make a Deal with Monty Hall

- One door hides a car, two hide goats.
- The contestant chooses any door.
- Monty always opens a different door with a goat. (He can do this because he knows where the car is.)
- The contestant is then allowed to switch doors if she wants.

What is the best strategy for winning a car?

(a) Switch  (b) Don’t switch  (c) It doesn’t matter
Board question: Monty Hall

Organize the Monty Hall problem into a tree and compute the probability of winning if you always switch.

Hint first break the game into a sequence of actions.
Independence

Events $A$ and $B$ are independent if the probability that one occurred is not affected by knowledge that the other occurred.

\[
\text{Independence} \iff P(A|B) = P(A) \quad (\text{provided } P(B) \neq 0)
\]
\[
\iff P(B|A) = P(B) \quad (\text{provided } P(A) \neq 0)
\]

(For any $A$ and $B$)

\[
\iff P(A \cap B) = P(A)P(B)
\]
Roll two dice and consider the following events

- $A = \text{‘first die is 3’}$
- $B = \text{‘sum is 6’}$
- $C = \text{‘sum is 7’}$

$A$ is independent of

- (a) $B$ and $C$
- (b) $B$ alone
- (c) $C$ alone
- (d) Neither $B$ or $C$. 
Bayes’ Theorem

Also called Bayes’ Rule and Bayes’ Formula.

Allows you to find $P(A|B)$ from $P(B|A)$, i.e. to ‘invert’ conditional probabilities.

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Often compute the denominator $P(B)$ using the law of total probability.
Board Question: Evil Squirrels

Of the one million squirrels on MIT’s campus most are good-natured. But one hundred of them are pure evil! An enterprising student in Course 6 develops an “Evil Squirrel Alarm” which she offers to sell to MIT for a passing grade. MIT decides to test the reliability of the alarm by conducting trials.
When presented with an evil squirrel, the alarm goes off 99\% of the time.

When presented with a good-natured squirrel, the alarm goes off 1\% of the time.

(a) If a squirrel sets off the alarm, what is the probability that it is evil?

(b) Should MIT co-opt the patent rights and employ the system?
One solution
(This is a base rate fallacy problem)
We are given:

\[ P(\text{nice}) = 0.9999, \quad P(\text{evil}) = 0.0001 \text{ (base rate)} \]

\[ P(\text{alarm} | \text{nice}) = 0.01, \quad P(\text{alarm} | \text{evil}) = 0.99 \]

\[
P(\text{evil} | \text{alarm}) = \frac{P(\text{alarm} | \text{evil})P(\text{evil})}{P(\text{alarm})} \\
= \frac{P(\text{alarm} | \text{evil})P(\text{evil})}{P(\text{alarm} | \text{evil})P(\text{evil}) + P(\text{alarm} | \text{nice})P(\text{nice})} \\
= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.01)(0.9999)} \\
\approx 0.01
\]
Squirrels continued

Summary:

Probability a random test is correct  =  0.99

Probability a positive test is correct  ≈  0.01

These probabilities are not the same!

Alternative method of calculation:

<table>
<thead>
<tr>
<th></th>
<th>Evil</th>
<th>Nice</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Alarm</td>
<td>99</td>
<td>9999</td>
<td>10098</td>
</tr>
<tr>
<td>No alarm</td>
<td>1</td>
<td>989901</td>
<td>989902</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>999900</td>
<td>1000000</td>
</tr>
</tbody>
</table>
Table Question: Dice Game

1. The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other.
2. The Roller selects one of the Randomizer’s fists and covertly takes the die.
3. The Roller rolls the die in secret and reports the result to the table.

Given the reported number, what is the probability that the 6-sided die was chosen? (Find the probability for each possible reported number.)