18.034 Problem Set 9

Due Wednesday, May 10 in class.

1. This problem (taken from Hirsch, Smale, and Devaney) is about the system

\[ x' = (\cos x)(.1 \sin x - \sin y), \quad y' = (\cos y)(\sin x + .1 \sin y). \]

a) Find all the equilibrium points. Decide (as completely as you can) which ones are stable and which are unstable.

b) Explain why an orbit which meets the horizontal line \( y = (2k + 1)\pi/2 \) (for \( k \) any integer) has to remain on that line for all time. Describe as completely as you can the orbits on this line.

c) Same question for the vertical line \( x = (2j + 1)\pi/2 \).

Pause for exposition: suppose \( y(t) \) is a solution of the autonomous differential equation system \( y' = F(y) \) in \( \mathbb{R}^n \), and that \( y(t) \) is defined for all \( t \). The positive limit set for \( y \), written \( \omega(y) \), by definition consists of all points \( x \in \mathbb{R}^n \) with the following property: there is a sequence \( t_j \to \infty \) so that

\[ \lim_{j \to \infty} y(t_j) = x. \]

In the same way we can define the negative limit set \( \alpha(y) \). These are introduced on page 531 of the text, but with no precise definition.

d) Suppose \( (x(t), y(t)) \) is a solution of the differential equation above with \( x(0) = y(0) = \pi/4 \). Find the positive limit set \( \omega(x, y) \). (I’m looking for a very precise answer, but not necessarily a very precise proof: just try to figure out what seems to be happening, make some pictures, and formulate a reasonable guess.)

2. This problem is about the same system of differential equations as in problem 1. Consider the function

\[ V(x, y) = (\cos x)(\cos y). \]

a) Find all the critical points of \( V \) (points where both partial derivatives are zero). Which ones are local maxima? Which ones are local minima? Which ones are neither?

b) In some parts of the plane \( V \) is increasing along solution curves to the differential equation; in some parts it is constant; and in some parts it is decreasing. Say as completely as you can what happens there.

c) Your answers to (b) should say that \( V \) (or some very simple modification of it) can serve as a Lyapunov function for the differential equation in certain regions. Explain how that works, and what conclusions you can draw about stability of certain equilibrium points.
3. This problem is about the van der Pol system

\[ \begin{align*}
x' &= y - f(x), \\
y' &= -x
\end{align*} \]

described in the text on page 525. Here \( f \) is a function with a continuous derivative satisfying the requirements

1. \( f \) is odd—that is, \( f(x) = -f(-x) \);
2. there is a positive number \( a \) so that \( f \) is (strictly) negative on \((0, a)\) and (strictly) positive on \((a, \infty)\).

The text also asks that \( f(x) \to \infty \) as \( x \to \infty \), but that doesn’t seem to matter for the questions I want to ask. (If you look up the van der Pol equation, you may find \( f(x) = cx^3 - dx \), with \( c \) and \( d \) positive constants. You may also find it written as a single second-order equation for \( y \); that fits better with some important applications to electrical circuits.)

a) Prove that \((0, 0)\) is the only equilibrium point.

b) Prove that the function \( x^2 + y^2 \) is strictly increasing along solutions for \( 0 < |x| < a \), and strictly decreasing along solutions for \(|x| > a \).

c) Prove that the equilibrium point \((0, 0)\) cannot be a positive limit point of any solution. Hint: if

\[ \lim_{j \to \infty} (x(t_j), y(t_j)) = (0, 0), \]

then \(|x(t_j)| < a\) for all large enough \( j \).

I want you to use the idea explained in Example 9.2.2 to prove that the van der Pol system above must have a limit cycle. Fix a positive number \( M \) so that

\[ |f(x) - 2x| < M, \quad (|x| \leq 2a). \]

Now consider the region \( R \) in the \((x, y)\) plane bounded by four curves: two arcs of the circle \( x^2 + y^2 = M^2 + 4aM + 8a^2 \), a segment of the line \( y - x = M \), and a segment of the line \( y - x = -M \). (The four “corners” of \( R \) are the points \((-2a, \pm M - 2a)\), and \((2a, \pm M + 2a)\).)

d) Prove that a solution crossing the circle \( x^2 + y^2 = c \) at a point where \(|x| \geq 2a\) must be crossing from outside to inside.

e) Prove that a solution crossing the line \( y - x = M \) where \(|x| \leq 2a\) must be crossing from above to below (in the direction of decreasing \( y - x \)). In exactly the same way, it follows that a solution crossing the line \( y - x = -M \) where \(|x| \leq 2a\) must be crossing from below to above.

f) Explain why a solution starting inside \( R \) must remain inside \( R \) for all positive time. Then explain how Theorem 9.2.1 proves that there must be a cycle inside \( R \).