

18.034 Problem Set 8

Due Monday, May 1 in class.

1. Suppose F is a function of n variables y_1, \dots, y_n , taking values in \mathbb{R}^m . Recall that F is said to *satisfy a Lipschitz condition* if there is a constant L so that

$$|F(y) - F(y')| \leq L|y - y'|$$

for any y and y' in the domain of F . (The absolute value signs indicate lengths of vectors. Assume that F is defined on some box, and that all the partial derivatives $\partial F/\partial y_j$ exist and are continuous on the box. (Each of these partial derivatives is a function on the box with values in \mathbb{R}^m .) Prove that F satisfies a Lipschitz condition on the box, by following this outline. You may use the fact that a continuous function on a box must be bounded; from this it follows that there is a constant A so that

$$|\partial F_i/\partial y_j| \leq A$$

everywhere in the box. (Here F_i is the i th coordinate function of F .)

a) Suppose $v \in \mathbb{R}^n$ is any vector. Define the directional derivative of F in the direction v to be

$$F_v = \sum_{j=1}^n v_j \frac{\partial F}{\partial y_j}.$$

(You may have seen this defined only for unit vectors v , or with some division by the length of v , but this definition is really what's needed here. Notice that F_v is a function on the box taking values in \mathbb{R}^m .) Prove that

$$|F_v| \leq mA|v|.$$

b) For y and y' in the box, define a new function g of one variable s , taking values in \mathbb{R}^m , by

$$g(s) = F(sy + (1-s)y').$$

Then g is defined on $[0, 1]$. Calculate $g'(s)$, and explain why it's continuous.

c) I showed in recitation that for an \mathbb{R}^m -valued function g defined on the unit interval and having a continuous derivative, we have

$$|g(1) - g(0)| \leq \sup_{0 \leq s \leq 1} |g'(s)|.$$

Use this fact to prove that F satisfies a Lipschitz condition on the box. (Be sure to say what estimate you get for the Lipschitz constant L .)

2. Text, page 451, number 15. In addition to writing down the differential equations, I'd like you to say as much as you can about solving them, in the case $k_1 = 1$, $k_2 = 2$, with initial conditions $x(0) = 2$, $y(0) = 1$, $z(0) = w(0) = 0$.

3. Some version of this problem was meant for the last exam, but the exam was long enough without it. It's about the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

a) Calculate the matrix exponential e^{tA} .

b) Write

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Find a solution of the driven differential equation

$$y' = Ay, \quad y(0) = 0.$$

4. This problem is about the differential equation (2) in Example 8.2.1 of the text.

a) List all the equilibrium points.

b) For each equilibrium point, use Theorem 8.2.1 to analyze the stability of the equilibrium.

c) By thinking about the physical system that the differential equation is supposed to model, find some function $E(x, y)$ that ought to describe the energy (kinetic plus potential) of the system. Use the given differential equations for x and y to show that E satisfies a differential equation $E' = -Ay^2$ (with A a positive constant). In particular, this means that E is always decreasing.

d) What does the fact that E is decreasing tell you about the shape of solution curves for the system? Can you see that it guarantees stability of some of the equilibrium points?

e) If energy is supposed to be conserved, why is E decreasing? Why is the rate of decrease proportional to y^2 ?