18.034 Problem Set 6

Due Monday, April 10 in class.

1. Consider the two vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and the two matrices

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

a) Prove that v_1 and v_2 are both eigenvectors of both M_1 and M_2. What are the eigenvalues?

b) Suppose a and b are any two real numbers. Write down a matrix M with the property that v_1 and v_2 are both eigenvectors of M, with eigenvalues a and b respectively.

c) Write down a system of two (constant coefficient first-order linear) differential equations with the property that the two sets of functions

$$x(t) = e^{at}, \qquad y(t) = e^{at}$$

and

$$x(t) = e^{bt}, \qquad y(t) = -e^{bt}$$

are both solutions.

d) For the differential equations you wrote in part (c), find explicit formulas for the solutions $x(t)$ and $y(t)$ satisfying the initial conditions

$$x(0) = x_0, \qquad y(0) = y_0.$$

2. This problem concerns the appearance of solution curves for the differential equations you wrote in 1(d). A "solution curve" means the path traced in the (x, y) plane by a solution $(x(t), y(t))$ as t varies from $-\infty$ to ∞.

a) As long as a and b are both non-zero, prove that the four half-lines

$$\{sv_1 \mid s > 0\}, \qquad \{sv_1 \mid s < 0\}, \qquad \{sv_2 \mid s > 0\}, \qquad \{sv_2 \mid s < 0\}$$

are all solution curves.

b) Suppose $a = 2$ and $b = 1$. In a picture showing the two lines $y = x$ and $y = -x$, draw several more solution curves of the equations in 1(d). (Hint: the solution curves off these two lines are parabolas.)

c) Same question as (b), but now with $a = 1$ and $b = 2$.

d) Same question as (b), but now with $a = 1$ and $b = -1$. What is the geometric shape of the solution curves now? What happens to the solutions as $t \to +\infty$? What happens as $t \to -\infty$?

3. You have a population of lab animals (varying with time t) consisting of $x(t)$ juvenilles, $y(t)$ adults, and $z(t)$ seniors.

The adults reproduce at rate b, which contributes a term $by(t)$ to the growth of the juvenille population.

The juvenilles turn into adults at a rate a, contributing a term $-ax(t)$ to the growth of the juvenille population, and a term $+ax(t)$ to the growth of the adult population.

The adults turn into seniors at a rate c, contributing $-cy(t)$ to the growth of the adult population and $+cy(t)$ to the growth of the senior population.

Finally, the seniors die at a rate d, contributing $-dz(t)$ to the growth of the senior population.

Assume that all four constants a, b, c, and d are strictly positive.

a) Write a system of three differential equations for the three functions $x(t)$, $y(t)$, and $z(t)$ reflecting the conditions described above.

b) Show that $-d$ is an eigenvalue of the system, and write down the corresponding solution of the differential equations.

c) Suppose that $b = c$. Show that 0 is an eigenvalue of the system, and write down the corresponding solution. Is this solution biologically reasonable?

d) Still supposing $b = c$, show that $-(a+b)$ is another eigenvalue, and write down the corresponding solution of the differential equations. Is this solution biologically reasonable?

e) Assume finally that $b > c$. What can you say about the last two eigenvalues of the system (other than $-d$)?