18.034 Problem Set 5

Due Wednesday, March 22 in class.

1. Suppose that $f(x)$ is a function periodic of period 2π that is piecewise continuous and bounded by a constant B:

$$f(x) \leq B, \qquad (x \in \mathbb{R}).$$

Write $\sum_{n=-\infty}^{\infty} c_n(f)e^{inx}$ for the Fourier exponential series of f (page 586 in the text).

a) Prove that

$$|c_n(f)| \leq B.$$

b) Prove that

$$\sum_{n=-\infty}^{\infty} |c_n(f)|^2 \leq B^2.$$

(This is a much stronger conclusion than in (a), and it is harder to prove. You can find hints in Theorem 7 on page 623 of the text.)

2. Suppose that $f(x)$ is a function periodic of period 2π, that f is continuous, and that f' is piecewise continuous. (Explanations will follow.) Prove that there is a constant A so that

$$|nc_n(f)| \leq A.$$

(Just as in problem 1, there is a much stronger statement you could make by working harder.)

Here are the explanations. To say that f' is piecewise continuous on an interval $[a, b]$ means that we can subdivide the interval into finitely many pieces

$$[a_0, a_1], \quad [a_1, a_2], \quad \ldots \quad [a_{n-1}, a_n], \qquad a_0 = a, \quad a_n = b$$

and find continuous functions g_i on $[a_{i-1}, a_i]$ with the property that

$$f'(x) = g_i(x) \qquad (x \in (a_{i-1}, a_i)).$$

We don't require $f'(a_i)$ to exist. If u and v are continuous functions on $[a, b]$ such that u' and v' are piecewise continuous, then integration by parts works:

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx.$$

You can use this formula in doing this problem; that's a hint.

3. Let f be the function which is periodic of period 2π, and which satisfies

$$f(\theta) = |\theta| \qquad (-\pi \leq \theta \leq \pi).$$

a) Sketch the graph of f on the interval $[-2\pi, 2\pi]$.

b) Calculate the nth exponential Fourier series coefficient

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

c) Assuming that the Fourier series of f converges to f (which it does—see Theorem 13 on page 629), show that

$$f(\theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)\theta}{(2m+1)^2}.$$

d) Prove that

$$\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \cdots.$$

e) Using just the formula in (d) and some algebraic cleverness, see if you can prove that

$$\pi^2/6 = 1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + \cdots.$$

4. The formulas in problem 3 are pretty lousy for actually calculating π: if you add up the first n terms, the error is something like $1/n$; so to get one more decimal place of accuracy requires using ten times as many terms in the series. Can you suggest anything similar to problem 3 that would lead to a faster-converging series related to π?

The next few problems concern delta functions. In the text on page 328 you can read about the Dirac delta function $\delta_0(t)$. Thought of as a driving force, $\delta_0(t)$ is a "unit impulse" at time $t = 0$. What that means informally is that the force is non-zero only at time zero, but its integral is nevertheless equal to 1. If you solve a driven differential equation

$$y'' + a(t)y' + b(t) = \delta_0(t),$$

what happens is that the solution turns a sharp corner at $t = 0$. The solution $y(t)$ is continuous, and it has a derivative everywhere except at 0. The derivative at 0 has a jump discontinuity:

$$\lim_{t \to 0^+} y'(t) = 1 + \lim_{t \to 0^-} y'(t).$$

This means that the impulse δ_0 instantaneously increases the velocity by 1 at time zero, without changing the position. For example, for the driven harmonic oscillator

$$y'' + y' = \delta_0(t), \qquad y(0) = 0,$$

the solutions are

$$y_A(t) = \begin{cases} (A - \frac{1}{2}) \sin t & \text{if } t \leq 0 \\ (A + \frac{1}{2}) \sin t & \text{if } t \geq 0. \end{cases}$$

The velocity is

$$y_A'(t) = \begin{cases} (A - \frac{1}{2}) \cos t & \text{if } t < 0 \\ (A + \frac{1}{2}) \cos t & \text{if } t > 0. \end{cases}$$

It's undefined when $t = 0$, which means that we can't easily write down an initial value problem to which y_A is a solution. A reasonable compromise is to declare that the velocity at time 0 is the average of its left and right limits, and to say that y_A is the solution of the initial value problem

$$y'' + y' = \delta_0(t), \qquad y(0) = 0, \qquad y'(0) = A.$$

4. Sketch the graph of the function $y_0(t)$ on the interval $[-2\pi, 2\pi]$.

5. Suppose we replace the single impulse $\delta_0(t)$ by a periodic impulse:

$$\delta_p(t) = \delta_0(t) + \delta_{2\pi}(t) + \delta_{-2\pi}(t) + \cdots.$$

Here $\delta_r(t)$ is a unit impulse at time r: so the effect of δ_p as a driving force is to increase the velocity by 1 every 2π seconds. Write y_p for the solution of the differential equation

$$y'' + y = \delta_p(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Describe the function y_p as completely as you can; sketch a graph of it on the interval $[-6\pi, 6\pi]$. (Hint: y_p is the same as y_0 on the interval $[-2\pi, 2\pi]$.)