18.034 Problem Set 4

Due Monday, March 6 in class.

1. Write down a second order linear differential equation with constant coefficients for which the function $y(t) = e^{-t/10}\cos t$ is a solution. What are all the solutions of this differential equation?

2. The differential equation you wrote for problem 1 looked like

$$y'' + ay' + by = 0.$$

This represents a harmonic oscillator with (slight) damping. Suppose now that you add a driving force $f(t) = \cos(\omega t)$, periodic with period $2\pi/\omega$. Find a periodic solution to the driven equation

$$y'' + ay' + by = \cos(\omega t).$$

(Let me emphasize that you are to solve this using the particular values of a and b from the first problem, but keeping ω as a parameter—a constant that we haven't fixed yet.)

3. The *amplitude* of a continuous periodic function is the largest value that it attains. Prove that your periodic solution from the second problem has amplitude equal to

$$\frac{1}{\sqrt{(\omega^2 - .99)^2 + .04}}.$$

4. Sketch a graph of the amplitude function from problem 3. What is its maximum value? For what values of ω is that maximum attained?

5. Think of the differential equation in problem 1 as a black box that you can "attach" to a driving force $f(t) = \cos(\omega t)$ with unknown frequency ω. The box detects whether or not ω is close to ± 1, by responding strongly. The response falls off by a factor of about two (below the maximum) if ω is outside the interval $[.5, 1.5]$ (and also outside $[-1.5, -.5]$). Can you explain how to modify these problems so that the response of the detector would fall off by a factor of ten if $\pm\omega$ is outside the interval $[.9, 1.1]$? (You don't need to carry this out in detail; just explain what kind of adjustments to make, and why they would work.)

6. Suppose that

$$y'' + a(t)y' + b(t)y = 0$$

is a second-order linear differential equation with non-constant coefficients, and that a and b are continuous functions of t. Suppose that y_1 and y_2 are two solutions of this differential equation. Define

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Show that $W(t)$ satisfies a certain first-order linear differential equation, and conclude that either $W(t) = 0$ for all t, or $W(t)$ is never zero.