18.02A PROBLEM SET 9 SOLUTIONS

1. (10 points) The Lasker Rink in the northeast corner of Central Park is a perfect circle having a radius of 50 meters. A skater moves counterclockwise around the edge of the rink at a speed of 5 meters per second. We'll put a coordinate system on the rink with origin at the center of the rink, the x-axis running east (toward Fifth Avenue) and the y-axis running north (toward Central Park North).

a) Suppose (x, y) is a point at the edge of the rink. Write down a non-zero tangent vector

$$\mathbf{T}(x, y) = A(x, y)\mathbf{i} + B(x, y)\mathbf{j}$$

to the rink at the point (x, y). (That is, write formulas for $A(x, y)$ and $B(x, y)$.)

There are lots of ways to do this. One way is to say that in order to be tangent to a curve $f(x, y) = c$, a vector has to be perpendicular to the gradient

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

The circle of radius 50 has equation $x^2 + y^2 = 2500$, so we can take $f(x, y) = x^2 + y^2$. Then the gradient is

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}.$$

A vector perpendicular to the gradient is

$$\mathbf{T}(x, y) = -2y\mathbf{i} + 2x\mathbf{j}.$$

(I got that by rotating the gradient through an angle of $\pi/2$.)

Another way to proceed is to write down parametric equations $x(s)$ and $y(s)$ for the edge of the rink and then to get a tangent vector by differentiating with respect to the parameter s:

$$\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}.$$

One natural parameter is the angle θ, leading to $x(\theta) = 50\cos\theta$, $y(\theta) = 50\sin\theta$. This gives the tangent vector

$$S(x(\theta), y(\theta)) = -50\sin\theta\mathbf{i} + 50\cos\theta\mathbf{j} = -y(\theta)\mathbf{i} + x(\theta)\mathbf{j}.$$

There are also more exotic choices, like

$$x(t) = 50(t^2 - 1)/(t^2 + 1), \qquad y(t) = 50(2t)/(t^2 + 1).$$

(This covers all of the circle except the point $(50, 0)$, as t runs over all real numbers). Such choices lead to more complicated formulas for tangent vectors, but they work.

b) Calculate the length of your tangent vector $\mathbf{T}(x, y)$.

The first one I wrote has length

$$\sqrt{(-2y)^2 + (2x)^2} = 2\sqrt{x^2 + y^2} = 2 \cdot 50 = 100.$$

c) The skater's velocity vector at $\mathbf{v}(x, y)$ must be some multiple of your tangent vector:

$$\mathbf{v}(x, y) = c(x, y)\mathbf{T}(x, y).$$

Calculate $c(x, y)$.

Since the tangent vector I wrote points counterclockwise, it corresponds to a velocity of 100 in the counterclockwise direction (with 100 being the length from part (b)). To get the skater's velocity, I need to reduce that to 5, which I can do by multiplying by

$$c(x, y) = 1/20.$$

The result is

$$\mathbf{v}(x, y) = (-y/10)\mathbf{i} + (x/10)\mathbf{j}.$$

d) Find a vector field F with the property that the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{R}$$

along the skater's path measures the distance that the skater has travelled to the west. (For example, over a path beginning at the northern extremity of the rink and continuing to the western extremity, the line integral ought to be 50 meters.)

If $\mathbf{F}(x, y) = -\mathbf{i}$ (a unit vector pointing due west), then $\mathbf{F} \cdot \mathbf{v}$ is always the westward component of $\mathbf{v}$. It follows that $\mathbf{F}\cdot d\mathbf{R}$ is the westward motion over the small piece $d\mathbf{R}$ of C, so the line integral of $\mathbf{F}$ over C gives the total westward motion.

e) Suppose the skater's path C begins at the point $(50\sqrt{3}/2, 25)$, and continues exactly halfway around the rink to the point $(-50\sqrt{3}/2, -25)$. Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{R}$$

using your formulas for F and $\mathbf{v} = d\mathbf{R}/dt$. (It's possible to get an answer without writing down any integrals precisely, but part of the problem is to write them down.)

Actually the outline indicated given here is a pretty slow way to compute the line integral; it's probably easier to ignore the first parts of the problem, simply writing down any old parametrization of C and computing. But I'll try to do what the problem suggests.

The idea is to use time as the parameter for the curve; that's what makes $d\mathbf{R}/dt$ equal to the velocity. Since the skater is travelling halfway around the rink, a distance of $\pi r = 50\pi$ meters, the time required is 10π seconds. The line integral is

$$\begin{aligned} &\int_0^{10\pi} \mathbf{F}(x(t), y(t))\cdot\mathbf{v}(x(t), y(t))dt \\ =&\int_0^{10\pi} -\mathbf{i}\cdot((-y(t)/10)\mathbf{i} + (x(t)/10)\mathbf{j})dt \\ =&\int_0^{10\pi} y(t)/10\, dt. \end{aligned}$$

To finish the problem this way without thinking, you need a formula for y as a function of t. For motion around a circle of radius r at angular velocity ω, the magic formulas are

$$x(t) = r\cos(\omega t + A), \qquad y(t) = r\sin(\omega t + A).$$

Here A is the angular position at time $t = 0$; in our case $\pi/6$ (which I calculated as the inverse tangent of y/x at time 0; that is, as the inverse tangent of $(25)/(50\sqrt{3}/2) = 1/\sqrt{3}$). Since the circumference of 100π is covered in time 20π seconds, the angular velocity ω is $2\pi/(20\pi) = 1/10$ radians per second. Therefore

$$y(t) = 50\sin(t/10 + \pi/6),$$

and

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_0^{10\pi} 5\sin(t/10+\pi/6)\,dt.\\ &= -50\cos(t/10+\pi/6)]_0^{10\pi} = -50(\cos(7\pi/6) - \cos(\pi/6))\\ &= -50(-\sqrt{3}/2 - \sqrt{3}/2) = 50\sqrt{3}.\end{aligned}$$

Remember that this line integral is supposed to measure distance travelled west; so its value should be just the *decrease* in the x coordinate over C. That's $50\sqrt{3}/2 - (-50\sqrt{3}/2)$, which agrees.

2. **(8 points) This problem is about whirlpools. A vector field F in the plane (possibly not defined at the origin) is called a *whirlpool* if its direction at every point is perpendicular to the radius. The most general whirlpool vector field looks like**

$$\mathbf{F}(x, y) = -f(r)y\mathbf{i} + f(r)x\mathbf{j},$$

with f a function of the radius $r = \sqrt{x^2 + y^2}$.

a) How much work is done by the whirlpool force field in moving counterclockwise around the circle of radius r centered at the origin? (The answer depends on $f(r)$.)

A whirlpool vector field always points along the circle, and has magnitude $r|f(r)|$. It points counterclockwise if f is positive and clockwise if f is negative. The work done by $\mathbf{F}$ in moving counterclockwise around the circle of radius r is therefore $2\pi r$ (the length of the circle) times $rf(r)$ (the magnitude of the force). This is $2\pi r^2 f(r)$.

b) The coordinates of F are

$$M(x, y) = -f(r)y, \qquad N(x, y) = f(r)x.$$

Suppose $f(r) = 1/r^2$, so that the whirlpool field has magnitude $1/r$. Calculate $\partial M/\partial y$ and $\partial N/\partial x$.

We have

$$M(x, y) = \frac{-y}{x^2 + y^2}, \qquad N(x, y) = \frac{x}{x^2 + y^2}.$$

Therefore

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{(x^2+y^2)(-1) - (2y)(-y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}.\\ \frac{\partial N}{\partial x} &= \frac{(x^2+y^2)(1) - (2x)(x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}.\end{aligned}$$

The thing to notice is that $\partial M/\partial y = \partial N/\partial x$.

c) Explain why part (a) shows that the only conservative whirlpool field is the zero field. Explain why part (b) shows that the whirlpool field with $f(r) = 1/r^2$ is conservative. [In the original problem, there were misprints here: a reference to "part (c)" and to "$f(r) = 1/r$."] Explain this apparent contradiction.

A vector field is conservative exactly when its integral around any closed path is zero. Part (a) shows that the integral of the whirlpool field around the (closed) circle of radius r is $2\pi r^2 f(r)$. This is zero for all r if and only if f is identically zero; that is, if and only if $\mathbf{F} = 0$.

Part (b) shows that the field with $f(r) = 1/r^2$ satisfies the derivative condition $\operatorname{curl} \mathbf{F} = 0$ to be conservative. Although this condition is *necessary* for being conservative (every conservative field has to satisfy it), the condition is *sufficient* only in regions with no holes. The region where $\mathbf{F}$ is defined (the whole plane except the origin) has a hole at the origin. So there is no contradiction.

3. (7 points) Problem 21.1(12) on page 757.

Since this was a straightforward calculation, I won't write out a solution.