1. (10 points) The Lasker Rink in the northeast corner of Central Park is a perfect circle having a radius of 50 meters. A skater moves counterclockwise around the edge of the rink at a speed of 5 meters per second. We’ll put a coordinate system on the rink with origin at the center of the rink, the $x$-axis running east (toward Fifth Avenue) and the $y$-axis running north (toward Central Park North).

a) Suppose $(x, y)$ is a point at the edge of the rink. Write down a non-zero tangent vector

$$T(x, y) = A(x, y)i + B(x, y)j$$

to the rink at the point $(x, y)$. (That is, write formulas for $A(x, y)$ and $B(x, y)$.)

There are lots of ways to do this. One way is to say that in order to be tangent to a curve $f(x, y) = c$, a vector has to be perpendicular to the gradient

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j.$$  

The circle of radius 50 has equation $x^2 + y^2 = 2500$, so we can take $f(x, y) = x^2 + y^2$. Then the gradient is

$$\nabla f = 2xi + 2yj.$$  

A vector perpendicular to the gradient is

$$T(x, y) = -2yi + 2xj.$$  

(I got that by rotating the gradient through an angle of $\pi/2$.)

Another way to proceed is to write down parametric equations $x(s)$ and $y(s)$ for the edge of the rink and then to get a tangent vector by differentiating with respect to the parameter $s$:

$$\frac{dx}{ds}i + \frac{dy}{ds}j.$$  

One natural parameter is the angle $\theta$, leading to $x(\theta) = 50\cos \theta$, $y(\theta) = 50\sin \theta$. This gives the tangent vector

$$S(x(\theta), y(\theta)) = -50\sin \theta i + 50\cos \theta j = -y(\theta)i + x(\theta)j.$$  

There are also more exotic choices, like

$$x(t) = 50(t^2 - 1)/(t^2 + 1), \quad y(t) = 50(2t)/(t^2 + 1).$$  

(This covers all of the circle except the point $(50, 0)$, as $t$ runs over all real numbers). Such choices lead to more complicated formulas for tangent vectors, but they work.

b) Calculate the length of your tangent vector $T(x, y)$.

The first one I wrote has length

$$\sqrt{(-2y)^2 + (2x)^2} = 2\sqrt{x^2 + y^2} = 2 \cdot 50 = 100.$$  

c) The skater’s velocity vector at $v(x, y)$ must be some multiple of your tangent vector:

$$v(x, y) = c(x, y)T(x, y).$$
Calculate $c(x,y)$.

Since the tangent vector I wrote points counterclockwise, it corresponds to a velocity of 100 in the counterclockwise direction (with 100 being the length from part (b)). To get the skater’s velocity, I need to reduce that to 5, which I can do by multiplying by

$$c(x,y) = \frac{1}{20}.$$ 

The result is

$$v(x,y) = (-y/10)i + (x/10)j.$$ 

d) Find a vector field $F$ with the property that the line integral

$$\int_C F \cdot dR$$

along the skater’s path measures the distance that the skater has travelled to the west. (For example, over a path beginning at the northern extremity of the rink and continuing to the western extremity, the line integral ought to be 50 meters.)

If $F(x,y) = -i$ (a unit vector pointing due west), then $F \cdot v$ is always the westward component of $v$. It follows that $F \cdot dR$ is the westward motion over the small piece $dR$ of $C$, so the line integral of $F$ over $C$ gives the total westward motion.

e) Suppose the skater’s path $C$ begins at the point $(50\sqrt{3}/2, 25)$, and continues exactly halfway around the rink to the point $(-50\sqrt{3}/2, -25)$. Evaluate the line integral

$$\int_C F \cdot dR$$

using your formulas for $F$ and $v = dR/dt$. (It’s possible to get an answer without writing down any integrals precisely, but part of the problem is to write them down.)

Actually the outline indicated given here is a pretty slow way to compute the line integral; it’s probably easier to ignore the first parts of the problem, simply writing down any old parametrization of $C$ and computing. But I’ll try to do what the problem suggests.

The idea is to use time as the parameter for the curve; that’s what makes $dR/dt$ equal to the velocity. Since the skater is travelling halfway around the rink, a distance of $\pi r = 50\pi$ meters, the time required is $10\pi$ seconds. The line integral is

$$\int_0^{10\pi} F(x(t), y(t)) \cdot v(x(t), y(t)) dt$$

$$= \int_0^{10\pi} -i \cdot ((-y(t)/10)i + (x(t)/10)j) dt$$

$$= \int_0^{10\pi} y(t)/10 dt.$$ 

To finish the problem this way without thinking, you need a formula for $y$ as a function of $t$. For motion around a circle of radius $r$ at angular velocity $\omega$, the magic formulas are

$$x(t) = r \cos(\omega t + A), \quad y(t) = r \sin(\omega t + A).$$
Here \( A \) is the angular position at time \( t = 0 \); in our case \( \pi/6 \) (which I calculated as the inverse tangent of \( y/x \) at time 0; that is, as the inverse tangent of \((25)/(50\sqrt{3}/2) = 1/\sqrt{3})\). Since the circumference of \(100\pi\) is covered in time \(20\pi\) seconds, the angular velocity \( \omega \) is \(2\pi/(20\pi) = 1/10\) radians per second. Therefore

\[
y(t) = 50\sin(t/10 + \pi/6),
\]

and

\[
\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{10\pi} 5\sin(t/10 + \pi/6) \, dt.
\]

\[
= -50\cos(t/10 + \pi/6)|_0^{10\pi} = -50(\cos(7\pi/6) - \cos(\pi/6))
\]

\[
= -50(-\sqrt{3}/2 - \sqrt{3}/2) = 50\sqrt{3}.
\]

Remember that this line integral is supposed to measure distance travelled west; so its value should be just the decrease in the \( x \) coordinate over \( C \). That’s \( 50\sqrt{3}/2 - (-50\sqrt{3}/2) \), which agrees.

2. (8 points) This problem is about whirlpools. A vector field \( \mathbf{F} \) in the plane (possibly not defined at the origin) is called a \textit{whirlpool} if its direction at every point is perpendicular to the radius. The most general whirlpool vector field looks like

\[
\mathbf{F}(x, y) = -f(r)y\mathbf{i} + f(r)x\mathbf{j},
\]

with \( f \) a function of the radius \( r = \sqrt{x^2 + y^2} \).

a) How much work is done by the whirlpool force field in moving counterclockwise around the circle of radius \( r \) centered at the origin? (The answer depends on \( f(r) \).)

A whirlpool vector field always points along the circle, and has magnitude \( r|f(r)| \). It points counterclockwise if \( f \) is positive and clockwise if \( f \) is negative. The work done by \( \mathbf{F} \) in moving counterclockwise around the circle of radius \( r \) is therefore \(2\pi r \) (the length of the circle) times \( r f(r) \) (the magnitude of the force). This is \(2\pi r^2 f(r) \).

b) The coordinates of \( \mathbf{F} \) are

\[
M(x, y) = -f(r)y, \quad N(x, y) = f(r)x.
\]

Suppose \( f(r) = 1/r^2 \), so that the whirlpool field has magnitude \( 1/r \). Calculate \( \partial M/\partial y \) and \( \partial N/\partial x \).

We have

\[
M(x, y) = \frac{-y}{x^2 + y^2}, \quad N(x, y) = \frac{x}{x^2 + y^2}.
\]

Therefore

\[
\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(-1) - (2y)(-y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.
\]

\[
\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)(1) - (2x)(x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.
\]

The thing to notice is that \( \partial M/\partial y = \partial N/\partial x \).

c) Explain why part (a) shows that the only conservative whirlpool field is the zero field. Explain why part (b) shows that the whirlpool field with \( f(r) = 1/r^2 \) is conservative. [In the original problem, there were misprints here: a reference to “part (c)” and to “\( f(r) = 1/r \).”] Explain this apparent contradiction.
A vector field is conservative exactly when its integral around any closed path is zero. Part (a) shows that the integral of the whirlpool field around the (closed) circle of radius $r$ is $2\pi r^2 f(r)$. This is zero for all $r$ if and only if $f$ is identically zero; that is, if and only if $F = 0$.

Part (b) shows that the field with $f(r) = 1/r^2$ satisfies the derivative condition $\text{curl } F = 0$ to be conservative. Although this condition is necessary for being conservative (every conservative field has to satisfy it), the condition is sufficient only in regions with no holes. The region where $F$ is defined (the whole plane except the origin) has a hole at the origin. So there is no contradiction.


Since this was a straightforward calculation, I won’t write out a solution.