18.02A PROBLEM SET 9

Due in recitation Tuesday, February 10

1. (10 points) The Lasker Rink in the northeast corner of Central Park is a perfect circle having a radius of 50 meters. A skater moves counterclockwise around the edge of the rink at a speed of 5 meters per second. We'll put a coordinate system on the rink with origin at the center of the rink, the x-axis running east (toward Fifth Avenue) and the y-axis running north (toward Central Park North).

a) Suppose (x, y) is a point at the edge of the rink. Write down a non-zero tangent vector

$$\mathbf{T}(x, y) = A(x, y)\mathbf{i} + B(x, y)\mathbf{j}$$

to the rink at the point (x, y). (That is, write formulas for $A(x, y)$ and $B(x, y)$.)

b) Calculate the length of your tangent vector $\mathbf{T}(x, y)$.

c) The skater's velocity vector at $\mathbf{v}(x, y)$ must be some multiple of your tangent vector:

$$\mathbf{v}(x, y) = c(x, y)\mathbf{T}(x, y).$$

Calculate $c(x, y)$.

d) Find a vector field $\mathbf{F}$ with the property that the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

along the skater's path measures the distance that the skater has travelled to the west. (For example, over a path beginning at the northern extremity of the rink and continuing to the western extremity, the line integral ought to be 50 meters.)

e) Suppose the skater's path C begins at the point $(50\sqrt{3}/2, 25)$, and continues exactly halfway around the rink to the point $(-50\sqrt{3}/2, -25)$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{R}$$

using your formulas for $\mathbf{F}$ and $\mathbf{v} = d\mathbf{R}/dt$. (It's possible to get an answer without writing down any integrals precisely, but part of the problem is to write them down.)

2. (8 points) This problem is about whirlpools. A vector field $\mathbf{F}$ in the plane (possibly not defined at the origin) is called a *whirlpool* if its direction at every point is perpendicular to the radius. The most general whirlpool vector field looks like

$$\mathbf{F}(x, y) = -f(r)y\mathbf{i} + f(r)x\mathbf{j},$$

with f a function of the radius $r = \sqrt{x^2 + y^2}$.

a) How much work is done by the whirlpool force field in moving counterclockwise around the circle of radius r centered at the origin? (The answer depends on $f(r)$.)

b) The coordinates of $\mathbf{F}$ are

$$M(x, y) = -f(r)y, \qquad N(x, y) = f(r)x.$$

Suppose $f(r) = 1/r^2$, so that the whirlpool field has magnitude $1/r$. Calculate $\partial M/\partial y$ and $\partial N/\partial x$.

c) Explain why part (a) shows that the only conservative whirlpool field is the zero field. Explain why part (c) shows that the whirlpool field with $f(r) = 1/r$ is conservative. Explain this apparent contradiction.

3. (7 points) Problem 21.1(12) on page 757.