18.02A PROBLEM SET 13

Due in recitation Tuesday, March 16

1. (10 points) This problem concerns an analogue of the theory of conservative fields using surface integrals in place of line integrals.

a) Suppose that

$$\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$$

is a vector field in space, and that each of the functions M, N and P has continuous second partial derivatives. Show that $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$.

b) You might say (although no one does!) that a vector field $\mathbf{G}$ in space is *superficially conservative* if the flux of $\mathbf{G}$ through a surface S depends only on the boundary curve C of S. That is, if S and S' are two surfaces having the same boundary curve C, then

$$\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_{S'} \mathbf{G}\cdot d\mathbf{S}.$$

Show that if $\mathbf{G} = \operatorname{curl}\mathbf{F}$, then $\mathbf{G}$ is superficially conservative. (Hint: use Stokes' theorem.)

c) Show that if $\mathbf{G}$ is superficially conservative, and S_0 is a closed surface, then

$$\iint_{S_0} \mathbf{G}\cdot d\mathbf{S} = 0.$$

d) Show that if $\mathbf{G}$ is superficially conservative, and the first partial derivatives of its coordinates are continuous, then $\operatorname{div}\mathbf{G} = 0$.

e) Suppose

$$\mathbf{G} = (y-z)\mathbf{i} + (z-x)\mathbf{j} + (x-y)\mathbf{k}.$$

Find a vector field $\mathbf{F}$ with the property that $\mathbf{G} = \operatorname{curl}\mathbf{F}$.

2. (15 points) Let S be the cylinder

$$S = \{(x,y,z) \mid x^2+y^2 = 1, \quad 0 \le z \le 1\},$$

oriented with an outward-pointing normal. Let C_1 be the top circle

$$C_1 = \{(x,y,1) \mid x^2+y^2 = 1\},$$

oriented counterclockwise, and C_0 the bottom circle

$$C_0 = \{(x,y,0) \mid x^2+y^2 = 1\},$$

also oriented counterclockwise. Finally (to make a tin can) look at the top and bottom discs

$$T_1 = \{(x,y,1) \mid x^2+y^2 \le 1\},$$

$$T_0 = \{(x,y,0) \mid x^2+y^2 \le 1\}.$$

Orient T_1 with the upward normal $\mathbf{k}$, and T_0 with the downward normal $-\mathbf{k}$. Finally, let D be the inside of the can

$$D = \{(x,y,z) \mid x^2+y^2 \le 1, \quad 0 \le z \le 1\}.$$

Consider the two vector fields

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{G} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}.$$

a) Calculate the curl and the divergence of $\mathbf{F}$ and $\mathbf{G}$.

b) Calculate the flux of $\mathbf{F}$, $\mathbf{G}$, $\operatorname{curl}\mathbf{F}$, and $\operatorname{curl}\mathbf{G}$ through each of the three surfaces S, T_0, and T_1.

c) Calculate the integrals of $\mathbf{F}$, $\mathbf{G}$, $\operatorname{curl}\mathbf{F}$, and $\operatorname{curl}\mathbf{G}$ around each of the closed curves C_1 and C_2.

d) Calculate the integrals of $\operatorname{div}\mathbf{F}$ and $\operatorname{div}\mathbf{G}$ over the region D.

e) Stokes' theorem and the divergence theorems provide a lot of relationships among your various answers to parts (b), (c), and (d). Write down as many of them as you can think of, in the forms that would apply to any vector field $\mathbf{F}$. (This means I am looking for formulas like

$$\iint_{T_0} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -\oint_{C_0} \mathbf{F}\cdot d\mathbf{r}$$

instead of formulas with your particular functions written out.)

3. (5 points). For the vector fields $\mathbf{F}$ and $\mathbf{G}$ of problem 2,

a) Find a function f with grad f $= \mathbf{F}$. Is there another function f' with grad f' $= \mathbf{G}$?

b) Find a vector field $\mathbf{H}$ with $\operatorname{curl}\mathbf{H} = \mathbf{G}$. Is there another vector field $\mathbf{H}'$ with $\operatorname{curl}\mathbf{H}' = \mathbf{F}$?