18.02A PROBLEM SET 12
Due in recitation Tuesday, March 9

1. (10 points) This problem is about the unit ball B in $\mathbb{R}^3$:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}.$$

The ball is divided into a top half and a bottom half:

$$B^+ = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1, z \ge 0\}, \quad B^- = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1, z \le 0\}.$$

The two half balls are separated by a disk E in the x-y plane. (The E stands for "equator.") The top surface of the ball is

$$S^+ = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \ge 0\},$$

and the bottom surface is

$$S^- = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \le 0\}.$$

Orient each of the three surfaces S^-, T, and S^+ with a generally upward normal (positive z coordinate). Consider the vector field $\mathbf{F} = (z + 1)\mathbf{k}$.

a) Calculate the flux of $\mathbf{F}$ (upward) through each of the three surfaces S^-, T, and S^+.

b) Calculate the two integrals

$$\iiint_{B^+} \operatorname{div} \mathbf{F}\, dV, \qquad \iiint_{B^-} \operatorname{div} \mathbf{F}\, dV.$$

c) Explain how to use the divergence theorem as a check on your answers to (a) and (b).

2. (15 points) Suppose $u(x, y, z)$ is a function defined everywhere in space, and having at least two continuous derivatives. The *Laplacian of* u is by definition

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

a) Suppose that D is a region in space bounded by a closed surface S. Show that the flux of the gradient of u across S is equal to the integral of the Laplacian of u across D: that is,

$$\iint_S \nabla u \cdot d\mathbf{S} = \iiint_D L(u) dV.$$

b) Suppose now that we have found a function u with the property that the gradient of u on the unit sphere is equal to the outward unit normal to the sphere. Calculate the flux of ∇u outward through the unit sphere.

c) If u is a function with the property specified in (b), and B is the unit ball, show that

$$\iiint_B L(u) dV = 4\pi.$$

Explain why it follows that $L(u)$ must be at least as big as 3 somewhere in B.

d) Find a specific function u with the property in (b), and find a specific place in B where $L(u)$ is at least 3. (Hint: part of what you want is that the gradient of u should be perpendicular to the surface $x^2 + y^2 + z^2 = 1$. What is the gradient of a function perpendicular to?)

3. (10 points) A roller coaster is designed with the parametric equations

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = 1 - \cos(3t),$$

The force of gravity on a roller coaster car is

$$\mathbf{F}(x, y, z) = -mg\mathbf{k},$$

with m the mass of the car and g the acceleration of gravity.

a) Let C_1 be the curve followed by the roller coaster from $t = 0$ to $t = \pi$. Calculate the work

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

done by gravity on the car in traversing this path.

b) Same question, but now for the path C_2 followed from $t = \pi$ to $t = 2\pi$.

c) Now we'll move the roller coaster to a tiny asteroid, where the force of gravity is

$$\mathbf{E}(x, y, z) = -\frac{mg}{(z+1)^2}\mathbf{k}.$$

Calculate the work

$$\int_{C_1} \mathbf{E} \cdot d\mathbf{r}.$$

(Hint: if you write out the line integral and compute, this is a little bit unpleasant. There's an easier way.)

4. (10 points).

a) Suppose f is a differentiable function of one variable, and $\mathbf{F}(x, y, z) = f(z)\mathbf{k}$. Show that $\mathbf{F}$ must be conservative, and explain how to use f to find a potential function for $\mathbf{F}$.

b) Suppose $\mathbf{E}$ is the vector field

$$\mathbf{E}(x, y, z) = \frac{1}{\rho}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

(Here $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.) The vector field is not defined at the origin. Show that $\operatorname{curl} \mathbf{E} = 0$, and find a potential function for $\mathbf{E}$.