18.02A PROBLEM SET 12
Due in recitation Tuesday, March 9

1. (10 points) This problem is about the unit ball $B$ in $\mathbb{R}^3$:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}.$$  

The ball is divided into a top half and a bottom half:

$$B^+ = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \geq 0\}, \quad B^- = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \leq 0\}.$$  

The two half balls are separated by a disk $E$ in the $x$-$y$ plane. (The $E$ stands for “equator.”) The top surface of the ball is

$$S^+ = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\};$$  

and the bottom surface is

$$S^- = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \leq 0\}.$$  

Orient each of the three surfaces $S^-$, $T$, and $S^+$ with a generally upward normal (positive $z$ coordinate). Consider the vector field $F = (z + 1)k$.

a) Calculate the flux of $F$ (upward) through each of the three surfaces $S^-$, $T$, and $S^+$.

b) Calculate the two integrals

$$\iiint_{B^+} \nabla \cdot F \, dV, \quad \iiint_{B^-} \nabla \cdot F \, dV.$$  

c) Explain how to use the divergence theorem as a check on your answers to (a) and (b).

2. (15 points) Suppose $u(x, y, z)$ is a function defined everywhere in space, and having at least two continuous derivatives. The Laplacian of $u$ is by definition

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$  

a) Suppose that $D$ is a region in space bounded by a closed surface $S$. Show that the flux of the gradient of $u$ across $S$ is equal to the integral of the Laplacian of $u$ across $D$: that is,

$$\iint_S \nabla u \cdot dS = \iiint_D L(u) \, dV.$$  

b) Suppose now that we have found a function $u$ with the property that the gradient of $u$ on the unit sphere is equal to the outward unit normal to the sphere. Calculate the flux of $\nabla u$ outward through the unit sphere.

c) If $u$ is a function with the property specified in (b), and $B$ is the unit ball, show that

$$\iiint_B L(u) \, dV = 4\pi.$$  

Explain why it follows that $L(u)$ must be at least as big as 3 somewhere in $B$. 
d) Find a specific function $u$ with the property in (b), and find a specific place in $B$ where $L(u)$ is at least 3. (Hint: part of what you want is that the gradient of $u$ should be perpendicular to the surface $x^2 + y^2 + z^2 = 1$. What is the gradient of a function perpendicular to?)

3. (10 points) A roller coaster is designed with the parametric equations

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = 1 - \cos(3t),$$

The force of gravity on a roller coaster car is

$$F(x, y, z) = -mg\mathbf{k},$$

with $m$ the mass of the car and $g$ the acceleration of gravity.

a) Let $C_1$ be the curve followed by the roller coaster from $t = 0$ to $t = \pi$. Calculate the work

$$\int_{C_1} F \cdot dr$$

done by gravity on the car in traversing this path.

b) Same question, but now for the path $C_2$ followed from $t = \pi$ to $t = 2\pi$.

c) Now we’ll move the roller coaster to a tiny asteroid, where the force of gravity is

$$E(x, y, z) = -\frac{mg}{(z + 1)^2}\mathbf{k}.$$ 

Calculate the work

$$\int_{C_1} E \cdot dr.$$ 

(Hint: if you write out the line integral and compute, this is a little bit unpleasant. There’s an easier way.)

4. (10 points).

a) Suppose $f$ is a differentiable function of one variable, and $F(x, y, z) = f(z)\mathbf{k}$. Show that $F$ must be conservative, and explain how to use $f$ to find a potential function for $F$.

b) Suppose $E$ is the vector field

$$E(x, y, z) = \frac{1}{\rho}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

(Here $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.) The vector field is not defined at the origin. Show that $\nabla \cdot E = 0$, and find a potential function for $E$. 