18.02A PROBLEM SET 11

Due in recitation Tuesday, February 24

1. (15 points) Let B be the unit ball in $\mathbb{R}^3$:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}.$$

This problem is about the function

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2},$$

which is defined on B.

a) What are the largest and smallest values of the function f?

b) The volume of B is equal to $4\pi/3$. Using just this fact and the answer to (a), what conclusion can you draw about the triple integral $\iiint_B f \, dV$? (Your answer should be something like "the integral is at most 11.")

c) There are at least three ways to calculate the integral: using rectangular coordinates, cylindrical coordinates, or spherical coordinates. Explain which of these ways ought to be the easiest.

d) Compute the integral in two different ways.

e) Let R be the unit disk in $\mathbb{R}^2$:

$$A = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

What is the geometric meaning of the double integral $\iint_R \sqrt{1 - x^2 - y^2} \, dA$?

f) Can you imagine a geometric meaning for the triple integral $\iiint_B f \, dV$?

2. (10 points) This problem is about the tetrahedron

$$T = \{(x, y, z) \mid x \geq 0, \ y \geq 0, \ z \geq 0, \ x + y + z \leq 1\}.$$

You may assume the fact that the volume of T is $1/6$.

a) If $P = (a, b, c)$ is a fixed point and $Q = (x, y, z)$ is a point in T, then the square of the distance from P to Q is $(x - a)^2 + (y - b)^2 + (z - c)^2$. Find the average value of the squared distance from P to points of T. (The answer will depend on a, b, and c.)

b) What point $P = (a, b, c)$ minimizes the average value of the squared distance to points of T?

c) Find the center of mass of the tetrahedron.

d) Can you make any insightful comments suggested by this problem?