

March 10, 2011

18.01 Problem Set 6
Due Wednesday, March 16, in recitation

Collaboration and discussion of problem sets is a good idea; you must write up your answers on your own, and you must answer question 0 of Part II.

The point of Part I is to push you to do some of the routine practice necessary to master this material. In order for that to work, you should write your solutions without consulting the solutions provided. There's no value for anyone in faithfully transcribing the typographical errors in those printed solutions.

Some of the questions in Part II are vague and open-ended, with no clear right answer. That's intentional; it's more like the form in which real mathematical questions appear. Say as much as you can.

Part I: 10 points

Notation for homework problems: "2.4/13" means Problem 13 at the end of section 2.4 in Simmons. "1A-3" means Exercise 1A-3 in Section E (Exercises) of the Supplementary Notes (solved in section S).

1. 2E-3, 8
2. 2F-1, 4.additional/119c
3. 2G-1ab, 3, 2.6/40
4. 3A-1ade, 2ahi, 3befg

Part II: 15 points

0. Write the names of all the people you consulted or with whom you collaborated and the resources you used, beyond the course text and notes and your instructors; or say "none" or "no consultation."
1. (10 points) This problem concerns the function $f(x) = \sqrt{1-x^2}$ on the interval $0 \leq x \leq 1$. I want to approximate this slightly complicated function by a straight line $y = ax + b$, where a and b are to be chosen. Usually when we talk about linear approximations in calculus, we are interested in x close to one special value x_0 ; but here I am considering all values of x equally. Since the function $f(x)$ is strictly decreasing, it's reasonable to think that a good approximation should be decreasing as well; so I will assume

a is strictly negative.

- a) Show that f is concave down on the interval $0 < x < 1$.
- b) Show that the derivative $f'(x)$ decreases from 0 toward $-\infty$ as x increases from 0 to 1.
- c) Show that the "error function"

$$e(x) = f(x) - ax - b$$

has exactly one critical point between 0 and 1, and that this critical point is a maximum.

- d) Show that the value of e at the critical point is $\sqrt{1+a^2} - b$.
- e) One way to choose a "best" straight line approximation to f is to require that the maximum value of $|e(x)|$ be as small as possible. Explain why this maximum value is the largest of the three numbers

$$|1 - b|, \quad |\sqrt{1+a^2} - b|, \quad |-a - b|.$$

- f) Show that if $-1 \leq a \leq 0$, then the maximum value of $|e|$ is $(\sqrt{1+a^2}+a)/2$, and find a similar formula for the case $-\infty < a \leq -1$.
- g) What is the best straight line approximation to f ? What is the maximum error of this approximation?
2. (5 points) Attached to every differentiation formula there is an integration formula: for example

$$d(u^n) = nu^{n-1}du \quad \leftrightarrow \quad \int nu^{n-1}du = u^n.$$

It's handy to adjust the constants on the right to make the integration formula more convenient, replacing n by $n+1$ and then dividing by n :

$$d(u^n) = nu^{n-1}du \quad \leftrightarrow \quad \int u^n du = u^{n+1}/(n+1) \quad (n \neq -1).$$

If we do this with the inverse trig functions \sin^{-1} and \cos^{-1} we get two formulas

$$d(\sin^{-1} u) = du/\sqrt{1-u^2} \quad \leftrightarrow \quad \int du/\sqrt{1-u^2} = \sin^{-1}(u). \quad (S)$$

$$d(\cos^{-1} u) = -du/\sqrt{1-u^2} \quad \leftrightarrow \quad \int du/\sqrt{1-u^2} = -\cos^{-1}(u). \quad (C)$$

The trouble is that this leads to two completely different formulas (S) and (C) for the same antiderivative. Explain this as carefully and completely as you can: for example, if only one of these formulas is right, say which one, and what's wrong with the other.