Midterm of Algebraic Geometry II

Apr. 6 noon -7 noon, 2020

Don’t forget to write down clearly your

Name: _____________________ and MIT ID: _____________________

Instructions.

• The exam contains 4 problems, adding up to 100 points.

• Please show necessary reasoning and/or computation. Any theorem or exercises proved during the class can be directly used without further arguments. Please don’t consult the internet.

• You should send the answer to cyxu@mit.edu by Apr. 7 noon.

• Stay safe and good luck with the exam!

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<th>Problem Number</th>
<th>Points</th>
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Problem 1 (20 points)
Let $C = \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C(1)$. Let $S = \text{Proj}_C(\mathcal{E})$ and $H = \mathcal{O}_S(1)$.

1. (10 points) Compute $H^0(S, H^\otimes m)$ for all $m \in \mathbb{Z}$.

2. (10 points) Compute $H^1(S, H^\otimes m)$ for all $m \gg 0$. (Can you compute it for all $m \in \mathbb{N}$?)
Problem 2 (20 points) Let $X$ be a proper scheme over a noetherian ring $A$, and $L$ a line bundle on $X$.

1. (10 points) $L$ is ample if and only if $L|_{X_{\text{red}}}$ is ample, where $X_{\text{red}}$ is the reduction of $X$.

2. (10 points) Let $f: Y \to X$ be a finite surjective morphism. Then $L$ is ample if and only if $f^*L$ is ample.
Problem 3 (10 points) Prove any non-proper smooth curve is affine.
Problem 4 (50 points)

Let $X$ be a smooth projective surface, $H$ be a very ample divisor on $X$ with $H^1(X, H) = 0$. Assume $C \cong \mathbb{P}^1 \subset X$ and $\deg(O(C)|_C) = -1$. Denote by $\deg(H|_C) = k$.

1. (10 points) Prove $H^1(X, O_X(H + jC))$ for any $0 \leq j \leq k$. (Hint: induction on $j$).

2. (15 points) Prove the linear system $|H + kC|$ is base point free, which induces a morphism $X \to X_1$. Verify the induced morphism is an isomorphism along $X \setminus C$ and the image of $C$ is a point.

3. (5 points) Let $X_0 \to X_1$ be the normalization, then show $X \to X_1$ factors through $f: X \to X_0$.

4. (10 points) Show $f(C)$ is a smooth point. (Hint: Let $\mathcal{I}$ be the ideal sheaf of $C$. Compute $H^0(C, \mathcal{I}^n/\mathcal{I}^{n+1})$ for all $n \in \mathbb{N}$. Apply Formal Function Theorem to $\varinjlim H^0(C, \mathcal{I}^n/\mathcal{I}^{n+1})$ to conclude.)

5. (10 points) Prove $X_0 = X_1$.

(The last two parts are more difficult.)
Problem 5 (5 points) Let $X$ be a scheme which is of finite type over a field $k$. Show that the following three conditions are equivalent.

1. $X \times_k \overline{k}$ is reduced, where $\overline{k}$ is the algebraic closure of $k$.
2. $X \times_k k_p$ is reduced, where $k_p$ is the perfect closure of $k$.
3. $X \times_k K$ is reduced for any field extension $K$ of $k$. 