18.700 Problem Set 1

1. (3 points) Consider the set of complex numbers

\[ G = \{a + bi \mid a, b \in \mathbb{Q}\}. \]

(The \( G \) stands for Gauss; these numbers might be called Gaussian rational numbers, although I don’t know if they actually are.) Is \( G \) a field (with the same addition and multiplication operations as in \( \mathbb{C} \))? For a question like this, you should either explain why all the axioms for a field are satisfied (you can assume that they hold for \( \mathbb{C} \)), or else explain why one of the axioms fails. A few sentences could be enough to write.

2. (3 points) Consider the set of complex numbers

\[ M = \{re^{\frac{\pi i}{2}} + se^{\pi i} \mid r, s \in \mathbb{Q}, \}. \]

Is \( M \) a field?

3. (3 points) Consider the set of complex numbers

\[ M = \{re^{2\pi i \theta} \mid r, \theta \in \mathbb{Q}. \}. \]

Is \( M \) a field?

For each of the following subsets of \( \mathbb{F}^3 \), determine whether it is a subspace of \( \mathbb{F}^3 \):

(a) \( \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\} \).
(b) \( \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\} \).
(c) \( \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\} \).
(d) \( \{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\} \).

Prove or give a counterexample: if \( U_1, U_2, W \) are subspaces of \( V \) such that

\[ V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W, \]

then \( U_1 = U_2 \).

Prove that if \((v_1, \ldots, v_n)\) spans \( V \), then so does the list

\[ (v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n) \]

obtained by subtracting from each vector (except the last one) the following vector.

Suppose \((v_1, \ldots, v_n)\) is linearly independent in \( V \) and \( w \in V \). Prove that if \((v_1 + w, \ldots, v_n + w)\) is linearly dependent, then \( w \in \text{span}(v_1, \ldots, v_n) \).