1. (15 points) Let $E$ consist of all real numbers of the form $a + b\sqrt{2}$, with $a$ and $b$ rational numbers.

a) Is $E$ closed under addition (of real numbers)?

b) Is $E$ closed under multiplication (of real numbers)?

c) Does $(5 + 4\sqrt{2})^{-1}$ belong to $E$?

   No

   Yes, it’s equal to ________.

d) Is $E$ a field?

2. (15 points) Give an example of a finite-dimensional vector space $V$ and three subspaces $U_1$, $U_2$, and $U_3$ for which

   \[
   \dim(U_1 + U_2 + U_3) \neq \dim U_1 + \dim U_2 + \dim U_3 \\
   - \dim U_1 \cap U_2 - \dim U_2 \cap U_3 - \dim U_1 \cap U_3 \\
   + \dim U_1 \cap U_2 \cap U_3.
   \]

   (Hint: there is an example with $\dim V = 2$.)
3 (20 points) Let \((v_1, v_2, v_3, v_4)\) be a linearly independent set of vectors of a vector space over \(\mathbb{R}\). Find out

a) Do there exist \(a, b \in \mathbb{R}\) such that

\[
\dim \text{span}(v_1 + 3v_2 - v_3, 2v_1 + av_2, b \cdot v_3, v_4) = 2.
\]

b) Find out all \(a, b \in \mathbb{R}\) such that

\[
\dim \text{span}(v_1 + 3v_2 - v_3, 2v_1 + av_2, b \cdot v_3, v_4) = 3.
\]
3 (20 points) Let $V$ be the real vector space consisting of all polynomials of degree less than or equal to three, and vanishing at $x = 1$. In the notation of the text, this is

$$V = \{ p \in P_3(\mathbb{R}) \mid p(1) = 0 \}.$$

Find a basis for $V$.

4 (30 points). Assume $(v_1, \ldots, v_n)$ is a list of vectors in $V$ such that $\dim \text{span}(v_1, \ldots, v_n) = k$. Let $w$ be another vector in $V$.

a) Find an example of $v_1, \ldots, v_n$ and $w$ such that

$$\dim \text{span}(v_1 + w, \ldots, v_n + w) < k.$$
b) Prove that 
\[ \dim \text{span}(v_1 + w, ..., v_n + w) \leq k + 1. \]

c) Prove that 
\[ \dim \text{span}(v_1 + w, ..., v_n + w) \geq k - 1. \]
(Hint: You can use the conclusion from b).)