In this note I use some terminologies about graphs without defining them. You can look them up at wikipedia: graph, vertex, edge, multiple edge, loop, bipartite graph, tree, degree, regular graph, adjacency matrix, walk, closed walk. In this note we never consider directed graphs and so the adjacency matrix will always be symmetric for us.

1. Just linear algebra

Many of the things described in this section is just the Frobenius–Perron theory specialized for our case. On the other hand, we will cheat. Our cheating is based on the fact that we will only work with symmetric matrices, and so we can do some shortcuts in the arguments.

We will use the fact many times that if $A$ is a $n \times n$ real symmetric matrix then there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors which we can choose\footnote{For the matrix $I$, any basis will consists of eigenvectors as every vectors are eigenvectors, but of course they won’t be orthonormal immediately.} to be orthonormal. Let $u_1, \ldots, u_n$ be the orthonormal eigenvectors belonging to $\lambda_1 \geq \cdots \geq \lambda_n$: we have $Au_i = \lambda_i u_i$, and $(u_i, u_j) = \delta_{ij}$.

Let us start with some elementary observations.

**Proposition 1.1.** If $G$ is a simple graph then the eigenvalues of its adjacency matrix $A$ satisfies $\sum \lambda_i = 0$ and $\sum \lambda_i^2 = 2e(G)$, where $e(G)$ denotes the number of edges of $G$. In general, $\sum \lambda_i^\ell$ counts the number of closed walks of length $\ell$.

**Proof.** Since $G$ has no loop we have

$$\sum \lambda_i = TrA = 0.$$  

Since $G$ has no multiple edges, the diagonal of $A^2$ consists of the degrees of $G$. Hence

$$\sum \lambda_i^2 = TrA^2 = \sum d_i = 2e(G).$$

The third statement also follows from the fact $TrA^\ell$ is nothing else than the number of closed walks of length $\ell$. \hfill \square
Using the next well-known statement, Proposition 1.2, one can refine the previous statement such a way that the number of walks of length $\ell$ between vertex $i$ and $j$ can be obtained as

$$\sum_k c_k(i, j)\lambda_k^\ell.$$ 

The constant $c_k(i, j) = u_{ik}u_{jk}$ if $u_k = (u_{1k}, u_{2k}, \ldots, u_{nk})$.

**Proposition 1.2.** Let $U = (u_1, \ldots, u_n)$ and $S = \text{diag}(\lambda_1, \ldots, \lambda_n)$ then

$$A = USU^T$$

or equivalently

$$A = \sum_{i=1}^n \lambda_i u_i u_i^T.$$ 

Consequently, we have

$$A^\ell = \sum_{i=1}^n \lambda_i^\ell u_i u_i^T.$$ 

**Proof.** First of all, note that $U^T = U^{-1}$ as the vectors $u_i$ are orthonormal. Let $B = USU^T$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the $i$'th coordinate is 1. Then

$$Bu_i = USU^T u_i = USE_i = (\lambda_1 u_1, \ldots, \lambda_n u_n) e_i = \lambda_i u_i = Au_i.$$ 

So $A$ and $B$ coincide on a basis, hence $A = B$. 

Let us turn to the study of the largest eigenvalue and its eigenvector.

**Proposition 1.3.** We have

$$\lambda_1 = \max_{||x||=1} x^T Ax = \max_{x \neq 0} \frac{x^T Ax}{||x||^2}.$$ 

Further, if for some vector $x$ we have $x^T Ax = \lambda_1 ||x||^2$, then $Ax = \lambda_1 x$.

**Proof.** Let us write $x$ in the basis of $u_1, \ldots, u_n$:

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n.$$ 

Then

$$||x||^2 = \sum_{i=1}^n \alpha_i^2.$$ 

and

$$x^T Ax = \sum_{i=1}^n \lambda_i \alpha_i^2.$$
From this we immediately see that
\[ x^T A x = \sum_{i=1}^{n} \lambda_i \alpha_i^2 \leq \lambda_1 \sum_{i=1}^{n} \alpha_i^2 = \lambda_1 ||x||^2. \]

On the other hand,
\[ u_1^T A u_1 = \lambda_1 ||u_1||^2. \]

Hence
\[ \lambda_1 = \max_{||x||=1} x^T A x = \max_{x \neq 0} \frac{x^T A x}{||x||^2}. \]

Now assume that we have \( x^T A x = \lambda_1 ||x||^2 \) for some vector \( x \). Assume that \( \lambda_1 = \cdots = \lambda_k > \lambda_{k+1} \geq \cdots \geq \lambda_n \), then in the above computation we only have equality if \( \alpha_{k+1} = \cdots = \alpha_n = 0 \). Hence
\[ x = \alpha_1 u_1 + \cdots + \alpha_k u_k, \]
and so
\[ A x = \lambda_1 x. \]

Proposition 1.4. Let \( A \) be a non-negative symmetric matrix. There exists a non-zero vector \( x = (x_1, \ldots, x_n) \) for which \( A x = \lambda_1 x \) and \( x_i \geq 0 \) for all \( i \).

Proof. Let \( u_1 = (u_{11}, u_{12}, \ldots, u_{1n}) \). Let us consider \( x = (|u_{11}|, |u_{12}|, \ldots, |u_{1n}|) \).
Then \( ||x|| = ||u_1|| = 1 \) and
\[ x^T A x \geq u_1^T A u_1 = \lambda_1. \]

Then \( x^T A x = \lambda_1 \) and by the previous proposition we have \( A x = \lambda_1 x \). Hence \( x \) satisfies the conditions.

Proposition 1.5. Let \( G \) be a connected graph, and let \( A \) be its adjacency matrix. Then
(a) If \( A x = \lambda_1 x \) and \( x \neq 0 \) then no entries of \( x \) is 0.
(b) The multiplicity of \( \lambda_1 \) is 1.
(c) If \( A x = \lambda_1 x \) and \( x \neq 0 \) then all entries of \( x \) have the same sign.
(d) If \( A x = \lambda x \) for some \( \lambda \) and \( x_i \geq 0 \), where \( x \neq 0 \) then \( \lambda = \lambda_1 \).

Proof. (a) Let \( x = (x_1, \ldots, x_n) \) and \( y = (|x_1|, \ldots, |x_n|) \). As before we have \( ||y|| = ||x|| \), and
\[ y^T A y \geq x^T A x = \lambda_1 ||x||^2 = \lambda_1 ||y||^2. \]

Hence
\[ A y = \lambda_1 y. \]
Let $H = \{ i \mid y_i = 0 \}$ and $V \setminus H = \{ i \mid y_i > 0 \}$. Assume for contradiction that $H$ is not empty. Note that $V \setminus H$ is not empty either as $x \neq 0$. On the other hand, there cannot be any edge between $H$ and $V \setminus H$: if $i \in H$ and $j \in V \setminus H$ and $(i, j) \in E(G)$, then

$$0 = \lambda_1 y_i = \sum_j a_{ij} y_j \geq y_j > 0$$

contradiction. But if there is no edge between $H$ and $V \setminus H$ then $G$ would be disconnected, which contradicts the condition of the proposition. So $H$ must be empty.

(b) Assume that $A\overline{x}_1 = \lambda_1 \overline{x}_1$ and $A\overline{x}_2 = \lambda_1 \overline{x}_2$, where $\overline{x}_1$ and $\overline{x}_2$ are independent eigenvectors. Note that by part (a), the entries of $\overline{x}_1$ is not 0, so we can choose a constant $c$ such that the first entry of $\overline{x} = \overline{x}_2 - c\overline{x}_1$ is 0. Note that $A\overline{x} = \lambda_1 \overline{x}$ and $\overline{x} \neq 0$ since $\overline{x}_1$ and $\overline{x}_2$ were independent. But then $\overline{x}$ contradicts part (a).

(c) If $A\overline{x} = \lambda_1 \overline{x}$, and $y = (|x_1|, \ldots, |x_n|)$ then we have seen before that $Ay = \lambda_1 y$. By part (b), we know that $\overline{x}$ and $y$ must be linearly dependent so $\overline{x} = y$ or $\overline{x} = -y$. Together with part (a), namely that there is no 0 entry, this proves our claim.

(d) Let $A\overline{u}_1 = \lambda_1 \overline{u}_1$. By part (c), all entries have the same sign, we can choose it to be positive by replacing $\overline{u}_1$ with $-\overline{u}_1$ if necessary. Assume for contradiction that $\lambda \neq \lambda_1$. Note that if $\lambda \neq \lambda_1$ then $\overline{x}$ and $\overline{u}_1$ are orthogonal, but this cannot happen as all entries of both $\overline{x}$ and $\overline{u}_1$ are non-negative, further they are positive for $\overline{u}_1$, and $\overline{x} \neq 0$. This contradiction proves that $\lambda = \lambda_1$.

So part (c) enables us to recognize the largest eigenvalue from its eigenvector: this is the only eigenvector consisting of only positive entries (or actually, entries of the same sign).

**Proposition 1.6.** (a) Let $H$ be a subgraph of $G$. Then $\lambda_1(H) \leq \lambda_1(G)$.

(b) Further, if $G$ is connected and $H$ is a proper subgraph then $\lambda_1(H) < \lambda_1(G)$.

**Proof.** (a) Let $\overline{x}$ be an eigenvector of length 1 of the adjacency matrix of $H$ such that it has only non-negative entries. Then

$$\lambda_1(H) = \overline{x}^T A(H) \overline{x} \leq \overline{x}^T A(G) \overline{x} \leq \max_{|z| = 1} \overline{z}^T A(G) \overline{z} = \lambda_1(G).$$
In the above computation, if $H$ has less number of vertices than $G$, then we complete $x$ with 0’s in the remaining vertices and we denote the obtained vector with $\bar{x}$ too in order to make sense for $\bar{x}^T A(G) \bar{x}$.

(b) Suppose for contradiction that $\lambda_1(H) = \lambda_1(G)$. Then we have equality everywhere in the above computation. In particular $\bar{x}^T A(G) \bar{x} = \lambda_1(G)$. This means that $\bar{x}$ is eigenvector of $A(G)$ too. Since $G$ is connected $\bar{x}$ must be a (or rather "the") vector with only positive entries by part (a) of the above proposition. But then $\bar{x}^T A(H) \bar{x} < \bar{x}^T A(G) \bar{x}$, a contradiction. □

**Proposition 1.7.** (a) We have $|\lambda_n| \leq \lambda_1$.

(b) Let $G$ be a connected graph and assume that $-\lambda_n = \lambda_1$. Then $G$ is bipartite.

(c) $G$ is a bipartite graph if and only if its spectrum is symmetric to 0.

**Proof.** (a) Let $x = (x_1, \ldots, x_n)$ be a unit eigenvector belonging to $\lambda_n$, and let $y = (|x_1|, \ldots, |x_n|)$. Then

$$|\lambda_n| = |x^T Ax| = |\sum a_{ij} x_i x_j| \leq \sum a_{ij} |x_i||x_j| = y^T Ay \leq \max_{i=1} |x^T A x| = \lambda_1.$$ (Another solution can be given based on the observation that $0 \leq Tr A^\ell = \sum \lambda_\ell$. If $|\lambda_n| > \lambda_1$ then for large enough odd $\ell$ we get that $\sum \lambda_\ell < 0$.)

(b) Since $\lambda_1 \geq \cdots \geq \lambda_n$, the condition can only hold if $\lambda_1 \geq 0 \geq \lambda_n$. Again let $x = (x_1, \ldots, x_n)$ be a unit eigenvector belonging to $\lambda_n$, and let $y = (|x_1|, \ldots, |x_n|)$. Then

$$\lambda_1 = |\lambda_n| = |x^T Ax| = |\sum a_{ij} x_i x_j| \leq \sum a_{ij} |x_i||x_j| = y^T Ay \leq \max_{i=1} |x^T A x| = \lambda_1.$$ We need to have equality everywhere. In particular, $y$ is the positive eigenvector belonging to $\lambda_1$, and all $a_{ij} x_i x_j$ have the same signs which can be only negative since $\lambda_n \leq 0$. Hence every edges must go between the sets $V^- = \{i \mid x_i < 0\}$ and $V^+ = \{i \mid x_i > 0\}$. This means that $G$ is bipartite.

(c) First of all, if $G$ is a bipartite graph with color classes $A$ and $B$ then the following is a linear bijection between the eigenspace of the eigenvalue $\lambda$ and the eigenspace of the eigenvalue $-\lambda$: if $A x = \lambda x$ then let $y$ be the vector which coincides with $x$ on $A$, and $-1$ times $x$ on $B$. It is easy to check that this will be an eigenvector belonging to $-\lambda$.

Next assume that the spectrum is symmetric to 0. We prove by induction on the number of vertices that $G$ is bipartite. Since the spectrum of the graph $G$ is the union of the spectrum of the components there must be a component $H$ with smallest eigenvalue $\lambda_n(H) = \lambda_n(G)$. Note that $\lambda_1(G) = |\lambda_n(G)| = |\lambda_n(H)| \leq \lambda_1(H) \leq \lambda_1(G)$ implies that $\lambda_1(H) = -\lambda_n(H)$. Since
$H$ is connected we get that $H$ is bipartite and its spectrum is symmetric to 0. Then the spectrum of $G \setminus H$ has to be also symmetric to 0. By induction $G \setminus H$ must be bipartite. Hence $G$ is bipartite.

**Proposition 1.8.** Let $\Delta$ be the maximum degree, and let $\bar{d}$ denote the average degree. Then

$$\max(\sqrt{\Delta}, \bar{d}) \leq \lambda_1 \leq \Delta.$$ 

**Proof.** Let $v = (1, 1, \ldots, 1)$. Then

$$\lambda_1 \geq \frac{v^T A v}{||v||^2} = \frac{2e(G)}{n} = \bar{d}.$$ 

If the largest degree is $\Delta$ then $G$ contains $K_{1,\Delta}$ as a subgraph. Hence

$$\lambda_1(G) \geq \lambda_1(K_{1,\Delta}) = \sqrt{\Delta}.$$ 

Finally, let $x$ be an eigenvector belonging to $\lambda_1$. Let $x_i$ be the entry with largest absolute value. Then

$$|\lambda_1||x_i| = |\sum_j a_{ij}x_j| \leq \sum_j a_{ij}|x_j| \leq \sum_j a_{ij}|x_i| \leq \Delta |x_i|.$$ 

Hence $\lambda_1 \leq \Delta$.

**Proposition 1.9.** Let $G$ be a $d$-regular graph. Then $\lambda_1 = d$ and its multiplicity is the number of components. Every eigenvector belonging to $d$ is constant on each component.

**Proof.** The first statement already follows from the previous propositions, but it also follows from the second statement so let us prove this statement. Let $x$ be an eigenvector belonging to $d$. We show that it is constant on a connected component. Let $H$ be a connected component, and let $c = \max_{i \in V(H)} x_i$, let $V_c = \{i \in V(H) \mid x_i = c\}$ and $V(H) \setminus V_c = \{i \in V(H) \mid x_i < c\}$. If $V(H) \setminus V_c$ were not empty then there exists an edge $(i, j) \in E(H)$ such that $i \in V_c$, $j \in V(H) \setminus V_c$. Then

$$dc = dx_i = \sum_{k \in N(i)} x_k \leq x_j + \sum_{k \in N(i), k \neq j} x_k < c + (d - 1)c = dc,$$

contradiction. So $x$ is constant on each component.
2. Expanders and Pseudorandom Graphs

In this section, $G$ always will be a $d$–regular graph. The goal of this section is to show how $\lambda_2$ and $\lambda_n$ measures the "randomness" of the graph.

Let $S, T \subseteq V(G)$. Let

$$e(S, T) = |\{(u, v) \in E(G) \mid u \in S, v \in T\}|.$$

Note that in the above definition, $S$ and $T$ are not necessarily disjoint. For instance, if $S = T$, then maybe a bit counter intuitively, $e(S, S)$ counts 2 times the number of edges induced by the set $S$. If $G$ were random then we would expect $e(S, T) \approx \frac{d|S||T|}{n}$.

**Theorem 2.1.** Let $G$ be a $d$–regular graph on $n$ vertices with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $S, T \subseteq V(G)$ such that $S \cup T = V(G)$ and $S \cap T = \emptyset$. Then

$$(d - \lambda_2)\frac{|S||T|}{n} \leq e(S, T) \leq (d - \lambda_n)\frac{|S||T|}{n}.$$

Before we start proving this theorem, we need a lemma.

**Lemma 2.2.** Let $A$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $u_1, \ldots, u_n$. Then

(a) $$\min_{\not= 0} \frac{x^T A x}{||x||^2} = \lambda_n.$$

(b) $$\max_{x \perp u_1} \frac{x^T A x}{||x||^2} = \lambda_2.$$

**Proof.** (a) Let $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$. Then

$$x^T A x = \sum_{i=1}^{n} \lambda_i \alpha_i^2 \geq \lambda_n \sum_{i=1}^{n} \alpha_i^2 = \lambda_n ||x||^2.$$

On the other hand, $u_1^T A u_1 = \lambda_n ||u_1||^2$. This proves part (a).

(b) Again let $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$. Since $x \perp u_1$, we have $\alpha_1 = (x, u_1) = 0$. Then

$$x^T A x = \sum_{i=1}^{n} \lambda_i \alpha_i^2 = \sum_{i=2}^{n} \lambda_i \alpha_i^2 \leq \lambda_2 \sum_{i=1}^{n} \alpha_i^2 = \lambda_2 ||x||^2.$$

On the other hand, $u_2^T A u_2 = \lambda_2 ||u_2||^2$. This proves part (b).
Proof of the theorem. Let $|S| = s$ and $|T| = t$. Let us consider the vector $\overline{x}$ which takes the value $t$ on the vertices of $S$ and the value $-s$ on the vertices of $T$. Then $\overline{x}$ is perpendicular to the all $\overline{1}$ vector, indeed $|S|t - |T|s = 0$. Note that $u_1 = \frac{1}{\sqrt{n}}\overline{1}$ so $\overline{x}$ is perpendicular to $u_1$. Let us consider
\[
\sum_{(i,j) \in E(G)} (x_i - x_j)^2 = d \sum_{i=1}^{n} x_i^2 - 2 \sum_{(i,j) \in E(G)} x_i x_j = d ||x||^2 - \overline{x}^T A \overline{x}.
\]
First of all, by the lemma we have
\[
(d - \lambda_2)||x||^2 \leq d||x||^2 - \overline{x}^T A \overline{x} \leq (d - \lambda_n)||x||^2.
\]
On the other hand,
\[
\sum_{(i,j) \in E(G)} (x_i - x_j)^2 = e(S, T)(t - (-s))^2 = e(S, T)(s + t)^2 = e(S, T)n^2.
\]
Note that
\[
||x||^2 = ts^2 + st^2 = st(s + t) = stn.
\]
Hence
\[
(d - \lambda_2)nst \leq e(S, T)n^2 \leq (d - \lambda_n)nst.
\]
In other words,
\[
(d - \lambda_2)\frac{st}{n} \leq e(S, T) \leq (d - \lambda_n)\frac{st}{n}.
\]

Definition 2.3. Let $S \subseteq V(G)$. The set of neighbors of $S$ is
\[
N(S) = \{ u \in V(G) \setminus S \mid \exists v \in S : (u, v) \in E(G) \}.
\]

Definition 2.4. A graph $G$ is called $(n, d, c)$-expander if $|V(G)| = n$, it is $d$–regular and
\[
|N(S)| \geq c|S|
\]
for every set $S$ satisfying $|S| \leq n/2$.

Intuitively, the larger the $c$, the better your network (your graph $G$) is: if you have a gossip then it spreads in a fast way in a good expander.

Theorem 2.5. A $d$–regular graph $G$ on $n$ vertices is an $(n, d, c)$–expander with $c = \frac{d - \lambda_2}{2d}$.

Proof. Let $S \subseteq V(G)$ with $|S| \leq n/2$. Let $T = V(G) \setminus S$, not that $|T| \geq n/2$. Then
\[
e(S, T) = e(S, N(S)) \leq d|N(S)|.$
By Theorem 2.1 we have
\[ e(S, T) \geq (d - \lambda_2) \frac{|S||T|}{n} \geq (d - \lambda_2)|S| \frac{1}{2}. \]

Hence
\[ d|N(S)| \geq \frac{d - \lambda_2}{2}|S|. \]

In other words,
\[ |N(S)| \geq \frac{d - \lambda_2}{2d}|S| = c|S|. \]

The quantity \( d - \lambda_2 \) is called the spectral gap.

Let’s see another corollary of Theorem 2.1. First we start with a definition.

**Definition 2.6.** A set \( S \subseteq V(G) \) is called an independent set if it induces the empty graph. (In other words, \( e(S, S) = 0 \).) The size of the largest independent set is denoted by \( \alpha(G) \).

**Theorem 2.7.** (Hoffman-Delsarte bound) Let \( G \) be a \( d \)-regular graph on \( n \) vertices with eigenvalues \( d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Then
\[ \alpha(G) \leq \frac{-\lambda_n n}{d - \lambda_n}. \]

**Proof.** Let \( S \) be the largest independent set, and \( T = V(G) \setminus S \). Then \( |S| = \alpha(G) \), and \( e(S, T) = d|S| = d\alpha(G) \). By Theorem 2.1 we have
\[ e(S, T) \leq (d - \lambda_n) \frac{|S||T|}{n}. \]

Hence
\[ d\alpha(G) \leq (d - \lambda_n) \frac{\alpha(G)(n - \alpha(G))}{n}. \]

By dividing by \( \alpha(G) \) and multiplying by \( n/(d - \lambda_n) \) we get that
\[ \frac{nd}{d - \lambda_n} \leq n - \alpha(G). \]

In other words,
\[ \alpha(G) \leq \frac{-\lambda_n n}{d - \lambda_n}. \]
The Hoffman-Delsarte bound is surprisingly good in a number of cases. Let’s see a bit strange application. A family $F = \{A_1, A_2, \ldots, A_m\}$ is called intersecting if $A_i \cap A_j \neq \emptyset$. Assume that $A_i \subseteq \{1, 2, \ldots, n\}$, and $|A_i| = k$ for all $i$. The question is the following: what’s the largest possible intersecting family of $k$-element subsets of $\{1, 2, \ldots, n\}$? If $k > n/2$ then any two $k$-subset is intersecting so the question is trivial. So let us assume that $k \leq n/2$. A good candidate for a large intersecting family is the family $F_1$ of those subsets which contains the element 1 (or actually any fixed element). Then $|F_1| = \binom{n-1}{k-1}$.

Erdős, Ko and Rado proved that this is indeed the largest possible size of an intersecting family of $k$-element subsets of $\{1, 2, \ldots, n\}$. Actually, they also proved that if $n > 2k$ then an intersecting family of size $\binom{n-1}{k-1}$ must contain a fixed element. For $n = 2k$ this is not true: any family will work where you don’t choose a set and its complement at the same time. Now we will only prove the weaker statement that $\binom{n-1}{k-1}$ is an upper bound (and actually we will cheat a bit as we cite a very non-trivial statement).

Let us define the following graph $G$: its vertex set consists of the $k$-element subsets of $\{1, 2, \ldots, n\}$ and two sets are joined by an edge if they are disjoint. This graph is called the Kneser$(n, k)$ graph. An independent set in this graph is exactly an intersecting family. The following theorem about its spectrum is non-trivial, its proof can be found in C. Godsil and G. Royle: Algebraic graph theory, page 200.

**Theorem 2.8.** The eigenvalues of the Kneser$(n, k)$ graph are

$$(-1)^i \binom{n-k-i}{k-i},$$

where $i = 0, \ldots, k$. The multiplicity of $\binom{n-k}{k}$ is 1, otherwise the multiplicity of $(-1)^i \binom{n-k-i}{k-i}$ is $\binom{n}{i} - \binom{n}{i-1}$ if $i \geq 1$.

Note that the Kneser-graph is $\binom{n-k}{k}$-regular, and according to the previous theorem, its smallest eigenvalue is $-\binom{n-k-1}{k-1}$. Then by the Hoffman-Delsarte bound we have

$$\alpha(\text{Kneser}(n, k)) \leq \frac{\binom{n-k-1}{k} \binom{n}{k}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}}.$$ 

Note that

$$\binom{n-k}{k} = \frac{n-k}{k} \binom{n-k-1}{k-1},$$

and so the denominator is

$$\left(\frac{n-k}{k} + 1\right) \binom{n-k-1}{k-1} = \frac{n}{k} \binom{n-k-1}{k-1}.$$
Hence

\[ \frac{\binom{n-k-1}{k-1} \frac{n}{k}}{\binom{n-k-1}{k-1} + \binom{n-k-1}{k-1}} = \frac{k}{n} \frac{n}{k} = \binom{n-1}{k-1}. \]

Voilá! Honestly this is probably the most complicated proof of the Erdős-Ko-Rado theorem, but there are some similar theorems where the only known proof goes through the eigenvalues of some similarly defined graph.

Now let us turn back to estimating \( e(S, T) \) for \( d \)-regular graphs. The following theorem is called the **expander mixing lemma**.

**Theorem 2.9** (Expander mixing lemma). Let \( G \) be a \( d \)-regular graph on \( n \) vertices with eigenvalues \( d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Let \( \lambda = \max(|\lambda_2|, \ldots, |\lambda_n|) = \max(|\lambda_2|, |\lambda_n|) \). Let \( S, T \subseteq V(G) \), then

\[
\left| e(S, T) - d \frac{|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.
\]

**Proof.** Let \( \chi_S \) and \( \chi_T \) be the characteristic vectors of the sets \( S \) and \( T \): so \( \chi_S(u) = 1 \) if \( u \in S \) and 0 otherwise. Observe that

\[
e(S, T) = \chi_S^T A \chi_T.
\]

Let us write up \( \chi_S \) and \( \chi_T \) in the orthonormal basis \( \mathbf{u}_1, \ldots, \mathbf{u}_n \) of eigenvectors. Note that we can choose \( \mathbf{u}_1 \) to be \( \frac{1}{\sqrt{n}} \mathbf{1} \). Let

\[
\chi_S = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i
\]

and

\[
\chi_T = \sum_{i=1}^{n} \beta_i \mathbf{u}_i.
\]

Then

\[
\chi_S^T A \chi_T = \sum_{i=1}^{n} \lambda_i \alpha_i \beta_i.
\]

Note that \( \alpha_1 = (\chi_S, \mathbf{u}_1) = \frac{|S|}{\sqrt{n}} \), and similarly \( \beta_1 = (\chi_T, \mathbf{u}_1) = \frac{|T|}{\sqrt{n}} \). Hence

\[
\lambda_1 \alpha_1 \beta_1 = d \frac{|S| |T|}{\sqrt{n} \sqrt{n}} = d \frac{|S||T|}{n}.
\]

Hence

\[
e(S, T) - d \frac{|S||T|}{n} = \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i.
\]
Then

\[ |e(S, T) - d\frac{|S||T|}{n}| = \left| \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i \right| \leq \lambda \sum_{i=2}^{n} |\alpha_i||\beta_i|.\]

Now let us apply a Cauchy-Schwartz inequality:

\[ \sum_{i=2}^{n} |\alpha_i||\beta_i| \leq \left( \sum_{i=2}^{n} |\alpha_i|^2 \right)^{1/2} \left( \sum_{i=2}^{n} |\beta_i|^2 \right)^{1/2}. \]

We will be a bit generous:

\[ \left( \sum_{i=2}^{n} |\alpha_i|^2 \right)^{1/2} \left( \sum_{i=2}^{n} |\beta_i|^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^{1/2} \left( \sum_{i=2}^{n} |\beta_i|^2 \right)^{1/2} = \]

\[ = ||\chi_S|| \cdot ||\chi_T|| = |S|^{1/2}|T|^{1/2}. \]

Hence

\[ |e(S, T) - d\frac{|S||T|}{n}| \leq \lambda \sqrt{|S||T|}. \]

If we were not generous at the last step then we could have proved the following stronger statement:

\[ |e(S, T) - d\frac{|S||T|}{n}| \leq \lambda \left( |S| - \frac{|S|^2}{n} \right)^{1/2} \left( |T| - \frac{|T|^2}{n} \right)^{1/2}. \]

We could have used that \( \alpha_1 = \frac{|S|}{\sqrt{n}} \) and \( \beta_1 = \frac{|T|}{\sqrt{n}} \).

**Remark 2.10.** A graph is called \((n, d, \lambda)\)-pseudorandom if it is a \(d\)-regular graph on \(n\) vertices with \(\max(|\lambda_2|, |\lambda_n|) \leq \lambda\). (Note that for bipartite \(d\)-regular graphs it is convenient to require that \(\lambda_2 \leq \lambda\), we will not do it though.) Many theorems which assert that "a random \(d\)-regular graph satisfies property \(P\) with probability 1" have an analogue that "an \((n, d, \lambda)\)-pseudorandom graph with \(\lambda \leq \ldots\) satisfies property \(P^n\)." Such an example is the following theorem due to F. Chung.

**Theorem 2.11.** \(^2\) Let \(G\) be an \((n, d, \lambda)\)-pseudorandom graph. Then the diameter of \(G\) is at most

\[ \left\lfloor \frac{\log(n-1)}{\log \left( \frac{4}{\lambda} \right)} \right\rfloor + 1. \]

\(^2\)We did not discuss this theorem at the lecture.
Proof. We need to prove that there exists an \( r \leq \lceil \log(n-1) \log(d) \rceil + 1 \) such that the distance between any vertices \( i \) and \( j \) is at most \( r \). In other words, there is a walk of length at most \( r \) starting at vertex \( i \) and ending at vertex \( j \). It means that we have to prove that \((A^r)_{ij} > 0\). On the other hand, we know that

\[ (A^r)_{ij} = \sum_{k=1}^{n} u_{ik} u_{jk} \lambda_k^r, \]

where \( u_k = (u_{1k}, \ldots, u_{nk}) \). As usual, \( u_1, \ldots, u_n \) is an orthonormal basis of eigenvectors: \( A u_i = \lambda_i u_i \), and \( u_1 = \frac{1}{\sqrt{n}} 1 \). Then

\[ u_{i1} u_{j1} \lambda_1^r = \frac{d}{n}. \]

So it is enough to prove that

\[ \left| \sum_{k=2}^{n} u_{ik} u_{jk} \lambda_k^r \right| < \frac{d}{n} \]

for some \( r \leq \lceil \log(n-1) \log(d) \rceil + 1 \).

The second inequality is a Cauchy-Schwartz. After that we used that the row(!) vectors of \( U = (u_1, \ldots, u_n) \) have length 1. This is true since the orthonormality of the column vectors implies the orthonormality of the row vectors. (Indeed, \( U \cdot U^T = I \) implies \( U^T \cdot U = I \).) Note that

\[ \lambda^r \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{1}{n} \right)^{1/2} = \lambda^r \left( 1 - \frac{1}{n} \right). \]

The second inequality is a Cauchy-Schwartz. After that we used that the row(!) vectors of \( U = (u_1, \ldots, u_n) \) have length 1. This is true since the orthonormality of the column vectors implies the orthonormality of the row vectors. (Indeed, \( U \cdot U^T = I \) implies \( U^T \cdot U = I \).) Note that

\[ \lambda^r \left( 1 - \frac{1}{n} \right) < \frac{d}{n}. \]

Indeed holds true for \( r = \lceil \log(n-1) \log(d) \rceil + 1 \).

It is clear from the previous theorems that the smaller the \( \lambda \), the better pseudorandom properties \( G \) have. Then the following question naturally arises: what's the best \( \lambda \) we can achieve? The complete graph on \( K_{d+1} \) has eigenvalues \( d, (-1)^{(d)} \), but the problem is that it is only one graph. What happens if we
require our graph to be large? The Alon-Boppana theorem asserts that in some sense $2\sqrt{d-1}$ is a threshold:

**Theorem 2.12 (Alon-Boppana).** Let $(G_n)$ be a sequence of $d$-regular graphs such that $|V(G_n)| \to \infty$. Then

$$\liminf_{n \to \infty} \lambda_2(G_n) \geq 2\sqrt{d-1}.$$ 

In other words, if $s < 2\sqrt{d-1}$ then there is only finitely many $d$–regular graphs for which $\lambda_2 \leq s$.

We will prove a slightly stronger statement due to Serre.

**Theorem 2.13 (Serre).** For every $\varepsilon > 0$, there exists a $c = c(\varepsilon, d)$ such that for any $d$–regular graph $G$, the number of eigenvalues $\lambda$ with $\lambda \geq (2-\varepsilon)\sqrt{d-1}$ is at least $c|V(G)|$.

Serre’s theorem indeed implies the Alon-Boppana theorem since for any $s < 2\sqrt{d-1}$ we choose $\varepsilon$ such that $s < (2-\varepsilon)\sqrt{d-1}$, then if $|V(G)| > 2/c(\varepsilon, d)$, we have at least two eigenvalues which are bigger then $s$ (one of them is $d$), so $\lambda_2(G) > s$. The following proof of Serre’s theorem is due to S. Cioaba.

**Proof.** The idea of the proof is that $p_{2k} = \sum_{i=1}^{n} \lambda_{2k}$ cannot be too small. Recall that $p_{2k}$ counts the number of closed walks of length $2k$. We will show that for any vertex $v$, the number of closed walks $W_{2k}(v)$ of length $2k$ starting and ending at $v$ is at least as large as the number of closed walks starting and ending at some root vertex of the infinite $d$–regular tree $T_d$.

Let us consider the following infinite $d$-regular tree, its vertices are labeled by the walks starting at the vertex $v$ which never steps immediately back to a vertex from where it came. Such walks are called non-backtracking walks. For instance, 149831 is such a walk, but 1494 is not a backtracking walk since after 9 we immediately stepped back to 4. We connect two non-backtracking walks in the tree if one of them is a one-step extension of the other. Note that every closed walk in the tree corresponds to a closed walk in the graph: for instance, 1, 14, 149, 14, 1 corresponds to 1, 4, 9, 4, 1. (In some sense, these are the "genuinely" closed walks.) On the other hand, there are closed walks in the graph $G$, like 149831, which are not closed anymore in the tree. Let $r_{2k}$ denote the number of closed closed walks from a given a root vertex in the infinite $d$–regular tree. So far we know that

$$p_{2k} = \sum_{v \in V(G)} W_{2k}(v) \geq nr_{2k}.$$
We would be able to determine $r_{2k}$ explicitly, but for our purposes, it is better to give a lower bound with which we can count easily. Such a lower bound is

$$r_{2k} \geq \frac{(2k)}{k+1} d(d - 1)^{k-1} > \frac{1}{(k+1)^2} (2\sqrt{d - 1})^{2k}.$$ 

The second inequality comes from Stirling’s formula, so we only need to understand the first inequality. Every closed walk in the tree can be encoded as follows: we write a 1 if we step down (so away from the root) and $-1$ if we step up (towards the root), additionally we choose a direction $d-1$ or $d$ ways if we step down. More precisely, we can choose our step in $d$ ways if we are in the root and $d-1$ ways otherwise. (The lower bound $d-1$ would be sufficient for us.) Note that we have to step down exactly $k$ times, and step up exactly $k$ times to get a closed walk. So the sequence of “directions” is at least $d(d-1)^{k-1}$. The sequence of $\pm 1$ has two conditions: (i) there must be exactly $k$ 1’s and exactly $k-1$’s, (ii) the sum of the first few elements cannot be negative (we cannot go higher than the root): $s_1 + s_2 + \cdots + s_i \geq 0$ for all $1 \leq i \leq 2k$, where $s_i = \pm 1$ according to the i-th step goes down or up. Such sequences are counted by the Catalan-numbers: $\frac{\binom{2k}{k}}{k+1}$. In this proof we simply accept this fact, later we will revisit this counting problem at $2 \times k$ standard Young tableauxs.

Now let us finish the proof by using the fact that

$$p_{2k} \geq \frac{n}{(k+1)^2} (2\sqrt{d - 1})^{2k}.$$
for every $k$. Let $m$ be the number of eigenvalues which are at least $(2 - \varepsilon)\sqrt{d - 1}$. Let us consider the sum

$$\sum_{i=1}^{n} (d + \lambda_i)^{2t},$$

where $t$ is a positive integer that we will choose later. Note that $0 \leq d + \lambda_i \leq 2d$, hence

$$\sum_{i=1}^{n} (d + \lambda_i)^{2t} \leq m(2d)^{2t} + (n - m)(d + (2 - \varepsilon)\sqrt{d - 1})^{2t}.$$

On the other hand, by the binomial theorem we have

$$\sum_{i=1}^{n} (d + \lambda_i)^{2t} = \sum_{i=1}^{n} \sum_{j=0}^{2t} \binom{2t}{j} d^j \lambda_i^{2t-j} = \sum_{j=0}^{2t} \binom{2t}{j} d^j \left( \sum_{i=1}^{n} \lambda_i^{2t-j} \right).$$

We know that $p_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$ if $k$ is odd and $p_{2k} \geq \frac{n}{(k+1)^2}(2\sqrt{d-1})^{2k}$. Hence

$$\sum_{j=0}^{2t} \binom{2t}{j} d^j \left( \sum_{i=1}^{n} \lambda_i^{2t-j} \right) \geq \sum_{j=0}^{t} \binom{2t}{2j} d^j \left( \sum_{i=1}^{n} \lambda_i^{2t-2j} \right) \geq \sum_{j=0}^{t} \binom{2t}{2j} d^j (2\sqrt{d-1})^{2t-2j} = \frac{n}{2(t+1)^2} \left( (d + 2\sqrt{d-1})^{2t} + (d - 2\sqrt{d-1})^{2t} \right) \geq \frac{n}{2(t+1)^2} (d + 2\sqrt{d-1})^{2t}.$$

Hence we have

$$m(2d)^{2t} + (n - m)(d + (2 - \varepsilon)\sqrt{d - 1})^{2t} \geq \frac{n}{2(t+1)^2} (d + 2\sqrt{d-1})^{2t}.$$

This means that

$$\frac{m}{n} \geq \frac{1}{2(t+1)^2} \frac{(d + 2\sqrt{d-1})^{2t} - (d + (2 - \varepsilon)\sqrt{d - 1})^{2t}}{(2d)^{2t} - (d + (2 - \varepsilon)\sqrt{d - 1})^{2t}}.$$

Note that

$$\left( \frac{d + 2\sqrt{d-1}}{d + (2 - \varepsilon)\sqrt{d - 1}} \right)^{2t}$$

grows much faster than $2(t+1)^2$, so we can choose a $t_0$ for which

$$\frac{1}{2(t_0+1)^2} (d + 2\sqrt{d-1})^{2t_0} - (d + (2 - \varepsilon)\sqrt{d - 1})^{2t_0} > 0,$$
then
\[ c(\varepsilon, d) = \frac{1}{2(t_0 + 1)^2} \left( (d + 2\sqrt{d - 1})^{2t_0} - (d + (2 - \varepsilon)\sqrt{d - 1})^{2t_0} \right) \]
satisfies the conditions of the theorem. \(\square\)

**Remark 2.14.** A \(d\)-regular non-bipartite graph \(G\) is called Ramanujan if \(\lambda_2, |\lambda_n| \leq 2\sqrt{d - 1}\). If \(G\) is bipartite then it is called Ramanujan if \(\lambda_2 \leq 2\sqrt{d - 1}\). It is known that for any \(d\) there exists infinitely many \(d\)-regular bipartite Ramanujan-graph, this is a result of A. Marcus, D. Spielman and N. Srivastava. On the other hand, if \(G\) is non-bipartite then our knowledge is much more limited: for \(d = p^\alpha + 1\), where \(p\) is a prime there exists construction for infinite family of \(d\)-regular Ramanujan-graphs. It is conjectured that a random \(d\)-regular graph is Ramanujan with positive probability independently of the number of vertices.

3. **Strongly regular graphs**

In this section we study strongly regular graphs, these are very special simple graphs. Strongly regular graphs are often very symmetric graphs and linear algebraic tools are particularly amenable to study them.

**Definition 3.1.** A graph \(G\) is a strongly regular graph with parameters \((n, d, a, b)\) if it has \(n\) vertices, \(d\)-regular, two adjacent vertices have exactly \(a\) common neighbors, and two non-adjacent vertices have exactly \(b\) common neighbors.

For instance, a 4-cycle is a strongly regular graph with parameters \((4, 2, 0, 2)\) while a 5-cycle is a strongly regular graph with parameters \((5, 2, 0, 1)\). Note that a \(k\)-cycle is never strongly regular if \(k \geq 6\). The Petersen-graph is a strongly regular graph with parameters \((10, 3, 0, 1)\).

![Figure 1. Petersen-graph as the Kneser(5,2) graph.](image)
In what follows we try to find necessary conditions for the parameters \((n, d, a, b)\) to enable the existence of a strongly regular graph with parameters \((n, d, a, b)\). The first one is very elementary.

**Proposition 3.2.** Let \(G\) be a strongly regular graph with parameters \((n, d, a, b)\). Then

\[
d(d - 1 - a) = (n - d - 1)b.
\]

**Proof.** Let \(u\) be a fixed vertex. Let us count the number of vertex pairs \((v_1, v_2)\) for which the following holds: \((u, v_1) \in E(G)\), \((v_1, v_2) \in E(G)\) and \((u, v_2) \notin E(G)\), and \(v_1, v_2 \neq u\).

We can choose \(v_1\) in \(d\) different ways, then we can choose \(v_2\) from the \(d\) neighbors of \(v_1\), but we cannot choose \(u\), and we cannot choose those \(a\) vertices which are connected to \(u\). So the number of these pairs are \(d(d - 1 - a)\).

On the other hand, we can choose \(v_2\) in \(n - d - 1\) different ways, as we cannot choose \(u\) and its neighbors. After choosing \(v_2\), we can choose \(v_1\) in \(b\) ways as \(u\) and \(v_1\) has \(b\) common neighbors.

Hence \(d(d - 1 - a) = (n - d - 1)b\). \(\Box\)

Next let us compute the eigenvalues and its multiplicities of a strongly regular graph. It will turn out that a strongly regular graph has only 3 different eigenvalues and the simple fact that the multiplicities of the eigenvalues must be non-negative integers imposes a very strong condition on the parameters \((n, d, a, b)\).

Recall that for a simple graph \(G\), the entries of \(A^2\) can be understood very easily. In the diagonal of \(A^2\) we have the degrees of the vertices, in our case, there will be \(d\) everywhere. On the other hand, for \(i \neq j\), \((A^2)_{ij}\) counts the number of common neighbors of vertex \(i\) and \(j\) which is \(a\) or \(b\) according to \(i\) and \(j\) are adjacent or not. So in \(A^2 + (b-a)A\) we have \(d's\) in the diagonal and \(b's\) everywhere else. Hence

\[
A^2 + (b-a)A - (d-b)I = bJ,
\]

where \(J\) is the all 1 matrix.

Now let us assume that \(Ax = \lambda x\), where \(x = (x_1, \ldots, x_n)\). Then

\[
(A^2 + (b-a)A - (d-b)I)x = (\lambda^2 + (b-a)\lambda - (d-b))x,
\]

while

\[
bJx = b(\sum_{i=1}^{n} x_i)1.
\]
Hence by comparing the $i$’th coordinates we get that

$$(\lambda^2 + (b-a)\lambda - (d-b))x_i = b(\sum_{i=1}^{n} x_i).$$

If $\lambda^2 + (b-a)\lambda - (d-b) \neq 0$, then all $x_i$ must be equal, and we simply get the usual eigenvector belonging to $d$. Otherwise, $\lambda^2 + (b-a)\lambda - (d-b)$ must be 0, hence

$$\lambda = \lambda_{\pm} = \frac{a - b \pm \sqrt{(a-b)^2 + 4(d-b)}}{2}.$$ 

It is easy to see that if $G$ is a disconnected strongly regular graph then it must be the disjoint union of some $K_{d+1}$. Since it is not really interesting, let us assume that $G$ is connected. Then we know that the multiplicity of the eigenvalue $d$ is exactly 1. Let $m_+$ and $m_-$ be the multiplicities of the other two eigenvalues. Since the number of eigenvalues is $n$, we know that

$$1 + m_+ + m_- = n.$$ 

We also know that $Tr A = 0$, so

$$0 = Tr A = 1 \cdot d + m_+ \lambda_+ + m_- \lambda_-.$$ 

From this we get that

$$m_{\pm} = \frac{1}{2} \left( n - 1 + \frac{2d + (n-1)(a-b)}{\sqrt{(a-b)^2 + 4(d-b)}} \right).$$ 

Let us summarize our results in a theorem.

**Theorem 3.3.** Let $G$ be a connected strongly regular graph with parameters $(n,d,a,b)$. Then its eigenvalues are $d$ with multiplicity 1, and

$$\lambda_{\pm} = \frac{a - b \pm \sqrt{(a-b)^2 + 4(d-b)}}{2}$$

with multiplicity

$$m_{\pm} = \frac{1}{2} \left( n - 1 + \frac{2d + (n-1)(a-b)}{\sqrt{(a-b)^2 + 4(d-b)}} \right).$$

As an example we can compute the eigenvalues of the Petersen-graph. Recall that this is a strongly regular graph with parameters $(10,3,0,1)$. Then its eigenvalues are $3,1$ and $-2$, where the multiplicities are $m_1 = 5$, $m_2 = 4$.

The condition that $m_{\pm}$ are non-negative integers is a surprisingly strong condition, this is called the *integrality condition*. As an application, let’s see which strongly regular graphs have parameters $(n,d,0,1)$. We have already
seen that the 5-cycle and the Petersen-graph are such graphs with \(d = 2\) and \(d = 3\). Actually, \(K_2\) is also such graph with \(d = 1\), but it’s a bit cheating since the fourth parameter hasn’t any meaning, not to mention \(K_1\) with \(d = 0\).

First of all, note that our first proposition implies that \(n = d^2 + 1\). Indeed, \(d(d-1-a) = (n-d-1)b\) with \(a = 0, b = 1\) immediately implies that \(n = d^2 + 1\).

**Theorem 3.4** (Hoffman–Singleton). Let \(G\) be a strongly regular graph with parameters \((d^2 + 1, d, 0, 1)\), where \(d \geq 2\). Then \(d \in \{2, 3, 7, 57\}\).

**Proof.** The eigenvalues of the graph \(G\) are \(\lambda = -1 \pm \sqrt{4d - 3}\) and its multiplicities are
\[
m_\pm = \frac{1}{2} \left( d^2 \mp \frac{2d - d^2}{\sqrt{4d - 3}} \right).
\]

If \(2d - d^2 = 0\), then \(d = 0\) or \(2\). (For \(d = 0\), the definition works, but we don’t consider it as a strongly regular graph. We simply excluded it by requiring \(d \geq 2\).) If \(2d - d^2 \neq 0\), then \(\sqrt{4d - 3}\) is a rational number. This can only happen if \(4d - 3\) is a perfect square. Hence \(4d - 3 = s^2\). Then
\[
m_\pm = \frac{1}{2} \left( \frac{s^2 + 3}{4} \right)^2 \mp \frac{2}{s} \left( \frac{s^2 + 3}{4} \right) - \frac{s^2 + 3}{4}.
\]

Hence
\[
m_+ = \frac{s^5 + s^4 + 6s^3 - 2s^2 + 9s - 15}{32s}.
\]

Since \(32m_+\) is an integer, we get that \(s \mid 15\). Hence \(s \in \{1, 3, 5, 15\}\). If \(s = 1\) then \(d = 1\) which we excluded. So \(s \in \{3, 5, 15\}\) whence \(d \in \{3, 7, 57\}\). Together with \(d = 2\) we get that \(d \in \{2, 3, 7, 57\}\).

**Remark 3.5.** One might wonder whether there is such a graph for \(d = 7\) and \(d = 57\). For \(d = 7\) there is such a graph: it is called the Hoffman-Singleton graph. It is the unique strongly regular graph with parameters \((50, 7, 0, 1)\) just as the Petersen-graph and the 5-cycle are the unique strongly regular graphs with parameters \((10, 3, 0, 1)\) and \((5, 2, 0, 1)\). It is not known whether there is a strongly regular graph with parameters \((3250, 57, 0, 1)\).

**Remark 3.6.** The following statement is true in general: the eigenvalues of a strongly regular graph are integers or the parameters of the strongly regular graph satisfies that \((n, d, a, b) = (4k + 1, 2k, k - 1, k)\) for some \(k\), the latter graphs are called conference graphs, for instance the 5–cycle is a conference...
graph. This statement can be proved by studying whether \( 2d + (n - 1)(a - b) \) is 0 or not.

**Theorem 3.7** (Lossers-Schwenk). *One cannot decompose \( K_{10} \) into three edge disjoint Petersen-graphs.*

**Proof.** Suppose for contradiction that we can decompose \( K_{10} \) into three edge disjoint Petersen-graphs. Let \( A_1, A_2 \) and \( A_3 \) be the adjacency matrices of the three Petersen-graphs. Then

\[
J - I = A_1 + A_2 + A_3.
\]

Note that \( A_1, A_2 \) and \( A_3 \) has a common eigenvector, namely the all-one vector. All other eigenvectors are orthogonal to this vector. In particular, we can consider the eigenspaces of \( A_1 \) and \( A_2 \) belonging to the eigenvalue 1. Let these eigenspaces be \( V_1 \) and \( V_2 \). Note that \( \dim V_1 = \dim V_2 = 5 \) as the multiplicity of the eigenvalue 1 is 5. We know that \( V_1, V_2 \subseteq \mathbb{1}^\perp \). Note that \( \mathbb{1}^\perp \) is a 9-dimensional vectorspace, so \( V_1 \) and \( V_2 \) must have a non-trivial intersection: let \( x \in V_1 \cap V_2 \).

Then

\[
A_3x = (J - I)x - A_1x - A_2x = 0 - x - x - x = -3x.
\]

But this is a contradiction since \(-3\) is not an eigenvalue of the Petersen-graph. \(\square\)

**Remark 3.8.** It is possible to pack two Petersen-graphs into \( K_{10} \). The above proof shows that the remaining edges form a 3–regular graph \( H \) with an eigenvalue \(-3\). This suggests that \( H \) should be a bipartite graph. This is indeed true, but one needs to prove first that \( H \) is connected. It is quite easy to prove it as the only disconnected 3-regular graph on 10 vertices is \( K_4 \cup K_3.3 \) (why?). It is not hard to show that \( H \) cannot be \( K_4 \cup K_3.3 \).

**Second proof.** Suppose that we can decompose \( K_{10} \) into 3 edge-disjoint Petersen-graphs. Let us color the edges of the three Petersen-graphs with blue, red and green colors in order to make it easier to refer to them. Let \( v \) be any vertex of \( K_{10} \) and let \( b_1, b_2, b_3 \) be the neighbors of \( v \) in the blue Petersen-graph. Similarly let \( r_1, r_2, r_3 \) and \( g_1, g_2, g_3 \) be the neighbors of \( v \) in the red and green Petersen–graphs.

For a moment, let’s put away the green Petersen–graph and let’s just concentrate on the bipartite graph induced by the vertices \( b_1, b_2, b_3 \) and \( r_1, r_2, r_3 \). Note that the edge \((v, r_1)\) is a red edge, so it’s not blue! This means that there must be exactly one blue path of length 2 between \( v \) and \( r_1 \). In other words, \( r_1 \) is connected by a blue edge to exactly one of the vertices of \( b_1, b_2, b_3 \). Similarly, \( r_2 \) and \( r_3 \) are connected by a blue edge to exactly one of the vertices of \( b_1, b_2, b_3 \). This means that there are exactly 3 blue edges between \( b_1, b_2, b_3 \)
and \( r_1, r_2, r_3 \). By repeating this argument to \( b_1, b_2, b_3 \), we find that there are exactly 3 red edges between \( b_1, b_2, b_3 \) and \( r_1, r_2, r_3 \). This means that there are exactly 3 green edges between \( b_1, b_2, b_3 \) and \( r_1, r_2, r_3 \).

Now let us consider the green Petersen–graph. If we delete the vertices \( v, g_1, g_2, g_3 \) from this graph, the green edges induce a 6-cycle on the remaining vertices. According to the previous paragraph, there is a cut of this 6-cycle which contains exactly 3 edges. But this is impossible: a cut of a 6-cycle always contains even number of edges! Indeed, if we walk around the 6-cycle we need to cross the cut even number of times to get back to the original side of the cut from where we started our walk. \( \Box \)