Additional notes on equiangular lines
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This is some additional text to Miniature 8 of Matousek’s Thirty-three miniatures. We follow the treatment of the book Algebraic graph theory by C. Godsil and G. Royle.

1. Equiangular lines

We say that distinct lines $\ell_1, \ell_2, \ldots, \ell_n$ of $\mathbb{R}^d$ going through the origin are equiangular if there exists some angle $\alpha$ such that for any $i \neq j$, the angle of $\ell_i$ and $\ell_j$ is $\alpha$. Clearly, if we chose a vector $x_i$ of length 1 on the line $\ell_i$, then $x_i^T x_j = \langle x_i, x_j \rangle = \pm \cos \alpha$. (Note that we consider $x_i$’s as column vectors.) For sake of simplicity let us introduce the notation $t = \cos \alpha$.

It will be convenient to introduce the matrices $X_i = x_i x_i^T$. Note that if $i \neq j$ then we have

$$\text{Tr}(X_i X_j) = \text{Tr}(x_i x_i^T x_j x_j^T) = \text{Tr}(x_i (x_i^T x_j) x_j^T) = \text{Tr}((x_i^T x_j) x_j^T x_i) = t^2$$

and

$$\text{Tr}(X_i X_i) = \text{Tr}(x_i x_i^T x_i x_i^T) = \text{Tr}(x_i (x_i^T x_i) x_i^T) = \text{Tr}(x_i x_i^T) = \text{Tr}(x_i^T x_i) = 1.$$  

We can see the same way that $\text{Tr}(X_i) = \text{Tr}(x_i x_i^T) = \text{Tr}(x_i x_i^T) = 1$.

In particular if for some matrix $Y$ we have

$$Y = \sum_{i=1}^n c_i X_i$$

then

$$\text{Tr}(Y^2) = \text{Tr} \left( \left( \sum_{i=1}^n c_i X_i \right)^2 \right) = \sum_{i=1}^n c_i^2 + 2 \sum_{1 \leq i < j \leq n} c_i c_j t^2 = (1-t^2) \sum_{i=1}^n c_i^2 + t^2 \left( \sum_{i=1}^n c_i \right)^2.$$  

This will be one of our main equations, so we repeat it:

(1) $$\text{Tr}(Y^2) = \text{Tr} \left( \left( \sum_{i=1}^n c_i X_i \right)^2 \right) = (1-t^2) \sum_{i=1}^n c_i^2 + t^2 \left( \sum_{i=1}^n c_i \right)^2.$$  

Via this equation we will give another proof of the absolute bound given in Miniature 8 of Matousek’s Thirty-three miniatures.

**Theorem 1.1.** (Absolute bound.) If distinct lines $\ell_1, \ell_2, \ldots, \ell_n$ of $\mathbb{R}^d$ going through the origin are equiangular then $n \leq \binom{d+1}{2}$. 

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Proof. Note that the matrices $X_i = x_i x_i^T$ are symmetric matrices of size $d \times d$. We will show that $X_1, \ldots, X_n$ are linearly independent matrices in the vectorspace of symmetric matrices of size $d \times d$. Since the dimension of this vectorspace is $\binom{d+1}{2}$, this would immediately prove our result.

Now if $X_1, \ldots, X_n$ were not linearly independent then for some $c_1, \ldots, c_n$ not all 0, we have

$$0_d = \sum_{i=1}^{n} c_i X_i,$$

where $0_d$ denotes the all 0 matrix of size $d \times d$. Then by Equation (1) we have

$$0 = Tr(0_d^2) = (1 - t^2) \sum_{i=1}^{n} c_i^2 + t^2 \left( \sum_{i=1}^{n} c_i \right)^2.$$

But the latter expression is clearly positive as $t^2 < 1$ and $\sum_{i=1}^{n} c_i^2 > 0$. This contradiction proves that $X_1, \ldots, X_n$ are linearly independent, and so $n \leq \binom{d+1}{2}$. □

Let us introduce the matrix $A$ which has columns $x_1, x_2, \ldots, x_n$. Then

$$AA^T = \sum_{i=1}^{n} x_i x_i^T = \sum_{i=1}^{n} X_i.$$

On the other hand,

$$A^T A = I + tS,$$

where $S$ is $\pm 1$ matrix, except in the diagonal where it is 0, since $x_i^T x_j = s_{ij} \cos \alpha = s_{ij} t$. The matrix $S$ is called the Seidel-matrix.

The next theorem will be essential when we start to study the case of equality in the absolute bound.

**Theorem 1.2.** Assume that

$$I = \sum_{i=1}^{n} c_i X_i.$$

Then $c_i = \frac{d}{n}$, and

$$n = \frac{d(1 - t^2)}{1 - dt^2}.$$

Furthermore, the eigenvalues of $S$ are $\frac{n-d}{d}$ with multiplicity $d$, and $\frac{1}{t}$ with multiplicity $n - d$. If $n \neq 2d$ then $\frac{1}{t}$ is an integer. Moreover, if $n \neq 2d$ and $d \leq n - 2$ then $\frac{1}{t}$ is an odd integer.
Proof. Since $I = \sum_{i=1}^{n} c_i X_i$, by Equation (1) we have

$$d = \text{Tr}(I^2) = (1 - t^2) \sum_{i=1}^{n} c_i^2 + t^2 \left(\sum_{i=1}^{n} c_i\right)^2.$$ 

Note that

$$X_k = \sum_{i=1}^{n} c_i X_i X_k,$$

so

$$1 = \text{Tr}X_k = \sum_{i=1}^{n} c_i \text{Tr}(X_i X_k) = c_k + t^2 \sum_{i \neq k} c_i = (1 - t^2)c_k + t^2 \sum_{i=1}^{n} c_i.$$

This means that

$$c_k = \frac{1 - t^2 \sum_{i=1}^{n} c_i}{1 - t^2},$$

which means that $c_1 = c_2 = \cdots = c_n = c$. Then

$$d = \text{Tr}I = c \sum_{i=1}^{n} \text{Tr}X_i = nc.$$

Hence we have $c = \frac{d}{n}$. If we write it back to the previous equation we get that

$$\frac{d}{n} = \frac{1 - t^2 \sum_{i=1}^{n} c_i}{1 - t^2} = \frac{1 - t^2d}{1 - t^2},$$

which means that

$$n = \frac{d(1 - t^2)}{1 - dt^2}.$$ 

We have seen that $AA^T = \sum_{i=1}^{n} X_i$, this means that $AA^T = \frac{n}{d}I$. Note that $AA^T$ and $A^T A$ have the same eigenvalues except the 0’s. This means that the eigenvalues of $AA^T = I + tS$ are $\frac{d}{n}$ with multiplicity $d$ and 0 with multiplicity $n-d$. In other words, the eigenvalues of $S$ are $\frac{n-d}{d}$ with multiplicity $d$, and $\frac{-1}{t}$ with multiplicity $n-d$.

Since the minimal polynomial of $\frac{-1}{t}$ divides the characteristic polynomial of $S$, and a minimal polynomial with coefficients in $\mathbb{Q}$ cannot have multiple zeros\footnote{This is true for any field of characteristic 0 (Why? Hint: consider $P'$ if $P$ is the minimal polynomial.), and even for many fields with characteristic $p$ including the finite fields $\mathbb{F}_q$.} we get that the minimal polynomial of $\frac{-1}{t}$ has degree 1 or 2. (Alternative way: the minimal polynomial of $S$ has degree 2 since it is symmetric. The minimal polynomial of $\frac{-1}{t}$ divides the minimal polynomial of $S$.) If the minimal polynomial of $\frac{-1}{t}$ is of degree 2 then $\frac{-1}{t}$ and $\frac{n-d}{dt}$ are algebraic conjugates.
and the characteristic polynomial of $S$ is a power of their common minimal polynomial, which would mean that $n - d = d$. So if $n - d \neq d$ then $\frac{-1}{t}$ is an integer since it is algebraic integer and rational.

Finally, if $n \neq 2d$ and $d \leq n - 2$ then let us consider the matrix

$$M = \frac{1}{2}(J - I + S),$$

where $J$ is the all 1 matrix. This is a $0 - 1$ matrix with 1’s at exactly at the places where $S$ has 1’s. The eigenspace of $J$ belonging to the eigenvalue 0 has dimension $n - 1$. The eigenspace of $S$ belonging to the eigenvalue $\frac{-1}{t}$ has dimension $n - d$. If $d \leq n - 2$ then these two eigenspaces have to intersect non-trivially, i. e. there is an eigenvector $x$ such that $Jx = 0$ and $Sx = \frac{-1}{t}x$ and so

$$Mx = \frac{1}{2}(J - I + S)x = \frac{1}{2}\left(-1 + \frac{-1}{t}\right)x.$$

This means that $\frac{1}{2}\left(-1 + \frac{-1}{t}\right)$ is an algebraic integer, and since $\frac{-1}{t}$ is an integer, it must be odd.

Now we are ready to study the case of equality in the absolute bound.

**Theorem 1.3.** If distinct lines $\ell_1, \ell_2, \ldots, \ell_n$ of $\mathbb{R}^d$ going through the origin are equiangular and $n = \left(\frac{d+1}{2}\right)$ then $d = 2, 3$ or $d + 2$ is an odd perfect square.

**Proof.** If $n - d = d$ and $n = \left(\frac{d+1}{2}\right)$ then $d = 3$. Let us assume that $d \neq 3$, and so $n - d \neq d$. We have seen that $X_1, \ldots, X_n$ are linearly independent in the vectorspace of symmetric matrices of size $d \times d$. Then they form a basis, and so there are $c_1, \ldots, c_n$ such that

$$I = \sum_{i=1}^{n} c_iX_i.$$

Then by the previous theorem

$$\left(\frac{d+1}{2}\right) = n \leq \frac{d(1 - t^2)}{1 - dt^2}$$

which gives that $\frac{1}{t^2} = d + 2$. If $n - d \neq d$ then $\frac{-1}{t}$ is an integer, so $d + 2$ must be a perfect square.

Finally, if $n \neq 2d$ and $d > 2$ and then $d \leq \left(\frac{d+1}{2}\right) - 2 = n - 2$ so $\frac{1}{t}$ is an odd integer, so $d + 2$ must be an odd perfect square. □

For $d = 3$, the six main diagonals of an icosahedron form a system of equiangular lines. For $d = 7$, we can consider the following 28 vectors in $\mathbb{R}^8$: $x_{ij}$ is
the vector where the $i$ and $j$-th coordinates are $\frac{3}{\sqrt{24}}$, the other coordinates are $-1$, for instance

$$\begin{pmatrix}x_{35} = \frac{1}{\sqrt{24}}(-1, -1, 3, -1, 1, -1, -1)\end{pmatrix}.$$ 

Then $||x_{ij}|| = 1$ and $x_{ij}^T x_{km} = \pm \frac{1}{3}$. Seemingly, all vectors are in $\mathbb{R}^8$, but since all $x_{ij}$ are orthogonal to the all 1 vector, we can consider them as vectors in $\mathbb{R}^7$. So this $\binom{8}{2} = 28$ lines form a system of equiangular lines in $\mathbb{R}^7$.

For $d = 23$ we will construct a system of equiangular lines of size $\binom{24}{2} = 276$. Before we do it, we will introduce the so-called Witt-design.