Compositions and partitions

Definition 1.1. A sequence \((a_1, \ldots, a_k)\) of non-negative integers with \(\sum_{i=1}^k a_i = n\) is called a weak composition of \(n\). If all \(a_i\) are positive then we say it is a composition of \(n\). (Note that the order of numbers matters.)

Theorem 1.2. The number of weak compositions of \(n\) into \(k\) parts is \(\binom{n + k - 1}{k - 1}\).

Theorem 1.3. The number of compositions of \(n\) into \(k\) parts is \(\binom{n - 1}{k - 1}\).

Theorem 1.4. The number of all compositions of \(n\) is \(2^{n-1}\).

Definition 1.5. A partition of the set \([n] = \{1, 2, \ldots, n\}\) is a collection of non-empty blocks so that each element belongs to exactly one of these blocks. The number of partitions of \([n]\) to exactly \(k\) parts is denoted\(^1\) by \(S(n, k)\) or \(\{n\}_k\), and these numbers are called Stirling numbers of second kind.

Theorem 1.6. For all positive integers \(n\) and \(k\) we have
\[
\binom{n}{k} = \binom{n-1}{k-1} + k \binom{n-1}{k}.
\]

Theorem 1.7. The number of all surjective functions from \(\{1, 2, \ldots, n\}\) to \(\{1, 2, \ldots, k\}\) is \(k! \{n\}_k\).

Theorem 1.8. For all real numbers \(x\) and all non-negative integers \(n\) we have
\[
x^n = \sum_{k=1}^{n} \binom{n}{k} x(x-1)\ldots(x-k+1).
\]

Definition 1.9. The number of all set partitions of \(n\) is denoted by \(B(n)\), and these numbers are called Bell-numbers.

Clearly, we have
\[
B(n) = \sum_{k=1}^{n} \binom{n}{k}.
\]

Theorem 1.10. For all non-negative integers \(n\) we have
\[
B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i).
\]

\(^1\)Bóna’s book uses \(S(n, k)\), I prefer \(\{n\}_k\).
2. Integer partitions

Definition 2.1. Let $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ be positive integers such that $\sum_{i=1}^{k} a_i = n$. Then $(a_1, \ldots, a_k)$ is called a partition of $n$. The number of partitions of $n$ is denoted by $p(n)$. The number of partitions of $n$ into exactly $k$ parts is denoted by $p_k(n)$.

Theorem 2.2. The number of partition of $n$ into at most $k$ is equal to the number of partitions of $n$ into parts not larger than $k$.

Proof. We will use Ferrers diagram. There is a natural bijection between the two sets: associate the conjugate partition to a partition. \qed

Theorem 2.3. The number of partition of $n$ into distinct odd parts is equal to that of all self-conjugate partitions.

We did not discuss the following two theorems mentioned in Bóna’s book, instead we studied the generating functions of partitions from Chapter 8.

Theorem 2.4. Let $q(n)$ be the number of partitions of $n$ in which each part is at least 2. Then $q(n) = p(n) - p(n-1)$ for all positive integers.

Theorem 2.5. Let $a = (a_1, \ldots, a_k)$ be a partition of the integer $n$, and let $m_i$ be the multiplicity of $i$ as a part of $a$. Then the number of set partition of $\{1, 2, \ldots, n\}$ that are of type $a$ is equal to

$$P_a = \frac{n!}{\prod_{i \geq 1} m_i!}.$$

Next let us understand the generating function of the sequence $(p_k(n))$.

Theorem 2.6. We have

$$\sum_{n=0}^{\infty} p_k(n)x^n = \prod_{i=1}^{k} \frac{1}{1 - x^i},$$

where $p_k(0) = 1$.

Proof. Note that

$$\prod_{i=1}^{k} \frac{1}{1 - x^i} = \prod_{i=1}^{k} (1 + x^i + x^{2i} + x^{3i} + \ldots).$$

If we expand this product, the coefficient of $x^n$ will come from the products of the form $x^{m_1 \cdot 1} \cdot x^{m_2 \cdot 2} \cdots x^{m_k \cdot k}$, where $m_1, \ldots, m_k \geq 0$ and $m_1 \cdot 1 + \cdots + m_k \cdot k = n$. Note that this naturally correspond to the partition in which we have $m_1$ 1’s, $m_2$ 2’s, ..., $m_k$ k’s and vice versa each partition naturally corresponds to such a term in the expansion. \qed
The same idea helps us to understand the generating function of all partitions.

**Theorem 2.7.** We have

\[ \sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}, \]

where \( p(0) = 1 \).

**Proof.** As before

\[ \prod_{i=1}^{\infty} \frac{1}{1 - x^i} = \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + x^{3i} + \ldots). \]

It might be scary to consider an infinite product, but observe that if you want to compute the coefficient of \( x^n \) then you always have to choose the term 1 from the terms \( 1 + x^i + x^{2i} + x^{3i} + \ldots \) when \( i \geq n + 1 \). Let us introduce the notation \( [x^n]f(x) \) for \( a_n \) if \( f(x) = \sum_n a_n x^n \). Then

\[ [x^n] \prod_{i=1}^{\infty} (1+x^i+x^{2i}+x^{3i}+\ldots) = [x^n] \prod_{i=1}^{n} (1+x^i+x^{2i}+x^{3i}+\ldots) = p_n(n) = p(n) \]

by the previous theorem and the fact that the largest part in a partition of \( n \) is at most \( n \). Hence

\[ \sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}. \]

\[ \square \]

One can think to generating functions \( \sum a_n x^n \) in two different ways:

(i) they are algebraic objects which form a ring, you can manipulate them algebraically, but you cannot plug any number (different form 0) into them,

(ii) they are analytic functions with some convergence radius.

The function \( \sum n! x^n \) is a good example for the difference between (i) and (ii). Since the convergence radius is 0 for this function, you will hardly be able to do anything with it analytically, but this is a completely eligible algebraic expression, a "prominent" element of a ring.

**Theorem 2.8.** Let \( p_o(n) \) be the number of partitions of \( n \) into odd parts. Let \( p_u(n) \) be the number of partitions of \( n \) into unequal parts. Then
\[
\sum_{n=0}^{\infty} p_o(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}.
\]

\[
\sum_{n=0}^{\infty} p_u(n)x^n = \prod_{i=1}^{\infty} (1 + x^i).
\]

\[
p_o(n) = p_u(n).
\]

**Proof.** The proof of part (a) and (b) goes as before. We only concentrate to part (c). Note that

\[
1 + x^i = \frac{1 - x^{2i}}{1 - x^i},
\]

hence

\[
\prod_{i=1}^{\infty} (1 + x^i) = \prod_{i=1}^{\infty} \frac{1 - x^{2i}}{1 - x^i} = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}.
\]

since the terms \(1 - x^{2k}\) will cancel from the denominator and the enumerator. Hence \(p_o(n) = p_u(n)\). □

**Second proof for part (c).** We will give a bijection between the set of partitions of \(n\) into odd parts and the set of partitions of \(n\) into unequal parts. The key ingredient of this bijection will be the observation that any number can be uniquely written into the form \(2^k(2t + 1)\), where \(k, t \geq 0\). So let \((\lambda_1, \ldots, \lambda_m)\) be a partition of \(n\) such that \(\lambda_1 > \cdots > \lambda_m\). Let \(\lambda_i = 2^{k_i}(2t_i + 1)\) and replace \(\lambda_i\) by \(2^{k_i}\) pieces of \(2t_i + 1\). Then clearly we obtained a partition of \(n\) into odd parts.

Now we show that we can decode the original partition. Let’s count the number of parts \(2t_i + 1\) in a partition of \(n\) into odd parts. Assume that there \(r_i\) pieces of \(2t_i + 1\). Then \(r_i\) can be uniquely written in base 2, i.e., there are unique \(s_1 > s_2 > \cdots > s_j\) such that \(r_i = 2^{s_1} + \cdots + 2^{s_j}\). Now replace the \(r_i\) pieces of \(2t_i + 1\) with elements \(2^{s_n}(2t_i + 1)\), where \(1 \leq n \leq j\).

Hence we gave a bijection between the set of partitions of \(n\) into odd parts and the set of partitions of \(n\) into unequal parts and so \(p_o(n) = p_u(n)\). □

An example for this proof is the following. Consider the partition \(8 + 6 + 4 + 3 + 1\), then \(8 = 2^3 \cdot 1, 6 = 2 \cdot 3, 4 = 2^2 \cdot 1, 3 = 3\) and \(1 = 1\). Hence the corresponding partition into odd parts will contain \(8 + 4 + 1 = 13\) pieces of 1’s \(2 + 1 = 3\) pieces of 3’s. And if you get the partition of 13 1’s and 3 pieces of
3’s then we know that we have to decompose 13 into 2-powers which can be uniquely done as $8 + 4 + 1$, and similarly $3 = 2 + 1$ so we get back the original partition.