**Chromatic polynomial**

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**Definition 1.1.** Let $G$ be a graph. A map $\varphi : V(G) \rightarrow \{1, 2, \ldots, q\}$ is a proper coloring with $q$ colors if $\varphi(u) \neq \varphi(v)$ whenever $(u, v) \in E(G)$.

The number of proper colorings of $G$ with $q$ colors is denoted by $ch(G, q)$.

**Proposition 1.2.** The function $ch(G, q)$ is polynomial in $q$.

*Proof.* If we use exactly $k$ colors then it corresponds to a decomposition of the vertex set into $k$ independent sets. So if $a_k(G)$ denotes the number of decompositions of the vertex set into $k$ independent sets then there are $a_k(G)q(q-1)\ldots(q-k+1)$ such proper colorings. Hence

$$ch(G, q) = \sum_{k=1}^{n} a_k(G)q(q-1)\ldots(q-k+1).$$

This is clearly a polynomial in $q$. \hfill $\Box$

**Proposition 1.3.** We have

$$ch(G, q) = ch(G - e, q) - ch(G/e, q),$$

where $G/e$ denotes the graph obtained from $G$ by contracting the edge $e$.

*Proof.* Let us consider the proper colorings of $G - e$. If $e = (u, v)$ then we can distinguish two cases: $u$ and $v$ get different colors then it is even a proper coloring of $G$. If $u$ and $v$ get the same color then it corresponds to a proper coloring of $G/e$. Hence

$$ch(G - e, q) = ch(G, q) + ch(G/e, q).$$

\hfill $\Box$

**Proposition 1.4.** For an $A \subseteq E(G)$ let $k(A)$ denote the number of components of the graph $(V(G), A)$. Then

$$ch(G, q) = \sum_{A \subseteq E(G)} (-1)^{|A|}q^{k(A)}.$$

*Proof.* Let $B$ the set of all colorings (not just the proper ones) of $V(G)$. Let $B_e$ the set of all colorings of $V(G)$, where the end vertices of $e$ get the same color. Then the number of proper colorings of $G$ is

$$ch(G, q) = |B \setminus \bigcup_{e \in E(G)} B_e|.$$
By inclusion-exclusion principle we have
\[ |B \setminus \bigcup_{e \in E(G)} B_e| = |B| - \sum_{e \in E(G)} |B_e| + \sum_{e_1, e_2 \in E(G)} |B_{e_1} \cap B_{e_2}| - \ldots. \]

Note that for some \( A \subseteq E(G) \) we have
\[ |\bigcap_{e \in A} B_e| = q^{k(A)}. \]

Hence
\[ ch(G, q) = \sum_{A \subseteq E(G)} (-1)^{|A|} q^{k(A)}. \]

**Proposition 1.5.** Let
\[ ch(G, q) = \sum_{k=0}^{n-1} (-1)^k c_{n-k} q^{n-k}. \]

Then \( c_i \geq 0. \)

**Proof.** We prove this claim by induction on the number of edges. For the empty graph \( O_n \) on \( n \) vertices we have \( ch(O_n, q) = q^n \), so the claim is true. Then from
\[ ch(G, q) = ch(G - e, q) - ch(G/e) \]
we have
\[ c_{n-k}(G) = c_{n-k}(G - e) + c_{n-k}(G/e). \]
(Note that \( G/e \) has \( n - 1 \) vertices!) Hence by induction
\[ c_{n-k}(G) = c_{n-k}(G - e) + c_{n-k}(G/e) \geq 0. \]

**Remark 1.6.** June Huh proved that the sequence \( (c_k(G))_{k=1}^n \) is log-concave, consequently unimodal. The coefficient \( c_{n-k}(G) \) has also a combinatorial meaning: it is the number of edge sets of size \( k \) not containing any broken cycle. A broken cycle is defined as follows: take any ordering of the edges, and a broken cycle is a cycle minus the highest index edge.

**Theorem 1.7** (A. Sokal). Let \( z \) be a zero of the chromatic polynomial \( ch(G, x) \). Then \( |z| \leq 8\Delta \), where \( \Delta \) is the maximum degree of \( G \).

**Remark 1.8.** Similar statements are true for the matching polynomial, characteristic polynomial, and the Laplacian polynomial, but those are much much simpler than Sokal’s theorem.
There are easier theorems for zero-free regions of the chromatic polynomial. For instance, from the identity
\[ ch(G, q) = \sum_{k=1}^{n} a_k(G)q(q-1)\ldots(q-k+1) \]
it is easy to prove that a real number \( z \) with \( z > n - 1 \) cannot be a zero of the chromatic polynomial. It is also easy to see from the recursion there is no zero between \((0, 1)\).

**Theorem 1.9** (R. Stanley). Let \( a(G) \) be the number of acyclic orientations. Then \( |ch(G, -1)| = a(G) \).

**Proof.** (Sketch) Note that \( |ch(G, -1)| = (-1)^n ch(G, -1) \) by the alternating sign property of the coefficients of the chromatic polynomial. By the recursion formula of the chromatic polynomial all we need to prove that \( a(G) = a(G - e) + a(G/e) \), then we can finish the proof by induction on the number of edges.

The equality \( a(G) = a(G - e) + a(G/e) \) holds true, because every acyclic orientation of \( G - e \) can be extended to an acyclic orientation of \( G \), and we can extended it in two different ways when contracting the end vertices of the edge \( e \), the resulting orientation is an acyclic orientation of \( G/e \). \(\square\)

**Proposition 1.10.** We have
\[ \sum_{S \subseteq V(G)} ch(G[S], x)ch(G[V \setminus S], y) = ch(G, x + y), \]
where \( G[S] \) is the induced subgraph of \( G \) on the set \( S \).

**Remark 1.11.** Similar identity holds true for the Laplacian-polynomial and the modified matching polynomial, \( M(G, x) = \sum_{k=0}^{\infty} (-1)^k m_k(G)x^{n-k} \).

**Proof.** It is enough to prove the identity for positive integer \( x \) and \( y \), then it is true for every \( x \) and \( y \). For positive integers \( x \) and \( y \), let us decompose the proper colorings of \( G \) with \( x + y \) colors according to the set \( S \) where we use the first \( x \) colors. Then we can color it in \( ch(G[S], x) \) ways, and we can color the remaining set \( V \setminus S \) in \( ch(G[V \setminus S], y) \) ways. From this the identity follows. \(\square\)