Chromatic number and bipartite graphs

Definition 1.1. A proper coloring of a graph $G$ is a coloring of its vertex set such a way that adjacent vertices get different colors. The smallest number of colors required for a proper coloring is the chromatic number of the graph, and is denoted by $\chi(G)$.

Proposition 1.2. (a) Let $G$ be a simple graph with largest degree $D$. Then $\chi(G) \leq D + 1$.
(b) Let $G$ be a graph on $n$ vertices with largest independent set of size $\alpha(G)$. Then $\chi(G) \geq \frac{n}{\alpha(G)}$.

Definition 1.3. A graph is called bipartite if its chromatic number is at most 2.

Theorem 1.4. A graph is bipartite if and only if it does not contain an odd cycle.

Let

\[ f(n) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even} \\ \frac{(n^2 - 1)}{4} & \text{if } n \text{ is odd} \end{cases} \]

Proposition 1.5. Let $G$ be a simple bipartite graph on $n$ vertices. Then it has at most $f(n)$ edges.

A much stronger statement is the following.

Theorem 1.6. Let $G$ be a simple graph on $n$ vertices with more than $f(n)$ edges. Then it contains a triangle.

An even more stronger statement is the following.

Theorem 1.7. Let $G$ be a simple graph on $2m$ vertices with more than $m^2 + 1$ edges. Then it contains at least $m$ triangles.

\[ ^1 \text{A subset } S \text{ of the vertex set is called an independent set if there is no edge between any two vertices of } S. \]