Chromatic number and bipartite graphs

Definition 1.1. A proper coloring of a graph G is a coloring of its vertex set such a way that adjacent vertices get different colors. The smallest number of colors required for a proper coloring is the chromatic number of the graph, and is denoted by $\chi(G)$.

Proposition 1.2. *(a) Let G be a simple graph with largest degree D. Then $\chi(G) \leq D + 1$.*

(b) Let G be a graph on n vertices with largest independent set[1] *of size $\alpha(G)$. Then $\chi(G) \geq \frac{n}{\alpha(G)}$.*

Definition 1.3. A graph is called bipartite if its chromatic number is at most 2.

Theorem 1.4. *A graph is bipartite if and only if it does not contain an odd cycle.*

Let

$$f(n) = \begin{cases} n^2/4 & \text{if } n \text{ is even} \\ (n^2-1)/4 & \text{if } n \text{ is odd} \end{cases}$$

Proposition 1.5. *Let G be a simple bipartite graph on n vertices. Then it has at most $f(n)$ edges.*

A much stronger statement is the following.

Theorem 1.6. *Let G be a simple graph on n vertices with more than $f(n)$ edges. Then it contains a triangle.*

An even more stronger statement is the following.

Theorem 1.7. *Let G be a simple graph on $2m$ vertices with more than $m^2 + 1$ edges. Then it contains at least m triangles.*

[1] A subset S of the vertex set is called an independent set if there is no edge between any two vertices of S.