In this note we consider the approximation of the Laplace single- and double-layer operators, which are given by

\[ S_0[\psi](x) = -\frac{1}{2\pi} \int_\Gamma \log(|x - y|) \psi(y) \, ds_y \quad \text{and} \quad (1) \]

\[ D_0[\psi](x) = \frac{1}{2\pi} \int_\Gamma \frac{(x - y) \cdot n_y}{|x - y|^2} \psi(y) \, ds_y, \quad (2) \]

respectively, where \( x \in \Gamma \). In order to evaluate this operators we first discretize the boundary \( \Gamma \) into line segments (elements) \( \Gamma_j, j = 1, \ldots, N \) (see Figure 1). We then proceed to expand the function \( \psi \) as

\[ \psi(y) \approx \sum_{j=1}^{N} \psi_j v_j(y), \quad (3) \]

in terms of piecewise constant basis functions \( v_j \) defined by

\[ v_j(y) = \begin{cases} 1 & \text{if } y \in \Gamma_j, \\ 0 & \text{if } y \not\in \Gamma_j. \end{cases} \]

The coefficients \( \psi_j \) in (3) are selected as \( \psi_j = \psi(x_j) \) where \( x_j \) is the midpoint of the linear segment \( \Gamma_j \). The integral operators are then approximated as follows

\[ S_0[\psi](x) \approx -\frac{1}{2\pi} \sum_{j=1}^{N} \psi_j \int_{\Gamma_j} \log(|x - y|) \, ds_y, \quad (4) \]

\[ D_0[\psi](x) \approx \frac{1}{2\pi} \sum_{j=1}^{N} \psi_j \int_{\Gamma_j} \frac{(x - y) \cdot n_y}{|x - y|^2} \, ds_y. \quad (5) \]

In order to evaluate the expressions in (4) and (5) we provide explicit expressions for the value of the integrals

\[ s_j(x) = \int_{\Gamma_j} \log(|x - y|) \, ds_y \quad \text{and} \quad d_j(x) = \int_{\Gamma_j} \frac{(x - y) \cdot n_y}{|x - y|^2} \, ds_y, \quad (6) \]
In order to do so we consider the line segment $\Gamma_j$ displayed in Figure 2, which is directed from $y_1$ to $y_2$. The quantity $d$ in that figure denotes the distance from the point $x$ to the line segment $\Gamma_j$ and is given by $d = \left| (y_1 - x) - \left( (y_1 - x) \cdot \left( \frac{y_2 - y_1}{L_j} \right) \right) \frac{y_2 - y_1}{L_j} \right|$ in terms the length $L_j = |y_2 - y_1|$. Using the variables defined in Figure 2 it easily follows that $r = |x - y| = d / \cos \theta$, $(x - y) \cdot n_y = -r \cos \theta$ and that the curve element is given by $d s_y = r d \theta / \cos \theta = d d \theta / \cos^2 \theta$. Parametrizing the integrals in (6) in terms of the angle $\theta$ we obtain

$$s_j(x) = \int_{\theta_1}^{\theta_2} \log \left( \frac{d}{\cos \theta} \right) \frac{d}{\cos^2 \theta} d \theta$$

$$= d \left\{ (\tan \theta_2 - \tan \theta_1) \log d + \theta_2 - \{1 + \log(\cos \theta_2)\} \tan \theta_2 - \theta_1 + \{1 + \log(\cos \theta_1)\} \tan \theta_1 \right\},$$

$$= r_2 \sin \theta_2 \{ \log(r_2) - 1 \} - r_1 \sin \theta_1 \{ \log(r_1) - 1 \} + d(\theta_2 - \theta_1), \quad (7)$$

$$d_j(x) = - \int_{\theta_1}^{\theta_2} d \theta = \theta_1 - \theta_2, \quad (8)$$

where $r_1 = |x - y_1|$ and $r_2 = |x - y_2|$. If $x_j$ corresponds to the midpoint of $\Gamma_j$, i.e., $x_j = (y_2 + y_1)/2$, we have $\theta_1 = -\pi/2$, $\theta_2 = \pi/2$, $r_1 = r_2 = L_{j/2}$ and $d = 0$. Therefore the expressions in (7) and (8) become

$$s_j(x_j) = L_j \log \left( \frac{L_{j/2}}{2} \right) - L_j,$$

$$d_j(x_j) = - \int_{\theta_1}^{\theta_2} d \theta = -\pi.$$

Consider now the following first- and second-kind integral equations

$$S_0[\psi] = f \quad \text{and} \quad \frac{\psi}{2} + D_0[\psi] = g \quad \text{on} \quad \Gamma,$$
respectively. Then the corresponding linear systems that result from application of the collocation method described above are given by

\[ S \psi = f \quad \text{and} \quad \left( \frac{1}{2} + D \right) \psi = g, \]

where the matrices \( S \in \mathbb{R}^{N \times N} \) and \( D \in \mathbb{R}^{N \times N} \) are defined by

\[ [S]_{i,j} = s_j(x_i), \quad [D]_{i,j} = \begin{cases} d_j(x_i), & i \neq j, \\ 0, & i = j, \end{cases} \quad i, j = 1, \ldots, N, \]

with \( x_i \) denoting the midpoint of the segment \( \Gamma_i \), \([f]_i = f(x_i), [g]_i = g(x_i)\) and \([\psi]_j = \psi_j\), where \( \psi_j \) is the approximation of \( \psi(x_j) \).