
Lecture 18

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Cambridge, November 9, 2017
Discrete Fourier transform (FFT)

• Suppose \( N \) is even. There is a connection between \( F_N \) and \( F_{N/2} \) that makes it possible to compute the FFT of order \( N \) as two FFT’s of order \( N/2 \).

• By repeating this process we can reduce the number of arithmetic operations to compute a DFT from \( O(N^2) \) to \( O(N \log N) \).

If \( N = 2m \), then

\[
F_{2m} P_{2m} = \begin{bmatrix}
F_m & D_m F_m \\
F_m & -D_m F_m
\end{bmatrix}
\]

where

\[
\omega_N = e^{-2\pi i / N},
\]

\[
D_m = \text{diag}\{1, \omega_N, \omega_N^2, \ldots, \omega_N^{m-1}\}
\]

\[
P_N = [e_1, e_3, \ldots, e_{N-1}, e_2, e_4, \ldots, e_N]
\]

and

\[
\{F_N\}_{jk} = \omega_N^{(j-1)(k-1)} = e^{-2\pi i (j-1)(k-1)/N}, \quad j, k = 1, \ldots, N.
\]
Fast Fourier transform (FFT)

• Suppose $N$ is even. There is a connection between $F_N$ and $F_{N/2}$ that makes it possible to compute the FFT of order $N$ as two FFT’s of order $N/2$.

• By repeating this process we can reduce the number of arithmetic operations to compute a DFT from $O(N^2)$ to $O(N \log N)$.

If $N = 2m$, then

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where

$$\omega_N = e^{-2\pi i/N},$$

$$D_m = \text{diag}\{1, \omega_N, \omega_N^2, \ldots, \omega_N^{m-1}\}$$

$$P_N = [e_1, e_3, \ldots, e_{N-1}, e_2, e_4, \ldots, e_N]$$

and

$$\{F_N\}_{jk} = \omega_N^{(j-1)(k-1)} = e^{-2\pi i(j-1)(k-1)/N}, \quad j, k = 1, \ldots, N.$$
Let \( y \in \mathbb{R}^{2^m} \). Then

\[
F_{2^m} y = F_{2^m} P_{2^m} \underbrace{P_{2^m}^T y}_{w=[w_1^T, w_2^T]} = \begin{bmatrix} F_m & D_m F_m \\ F_m & -D_m F_m \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} F_m w_1 + D_m F_m w_2 \\ F_m w_1 - D_m F_m w_2 \end{bmatrix} = \begin{bmatrix} q_1 + q_2 \\ q_1 - q_2 \end{bmatrix}
\]

where

\[
q_1 = F_m w_1 \quad \text{and} \quad q_2 = D_m F_m w_2.
\]

**Computational complexity** (special case \( N = 2^k \))

\( x_k \): number of arithmetic operations required to apply one FFT of order \( 2^k \).

\[
x_k = 2x_{k-1} + \gamma 2^{k-1}, \quad x_0 = 0
\]
Fast Fourier transform (FFT)

Let \( y \in \mathbb{R}^{2m} \). Then

\[
F_{2m} y = F_{2m} P_{2m} \underbrace{P_{2m}^T y}_{w = [w_1^T, w_2^T]} = \begin{bmatrix} F_m & D_m F_m \\ F_m & -D_m F_m \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} F_m w_1 + D_m F_m w_2 \\ F_m w_1 - D_m F_m w_2 \end{bmatrix} = \begin{bmatrix} q_1 + q_2 \\ q_1 - q_2 \end{bmatrix}
\]

where

\[
q_1 = F_m w_1 \quad \text{and} \quad q_2 = D_m F_m w_2.
\]

\[\text{Computational complexity} \quad \text{(special case } N = 2^k \text{)}\]

\( x_k \): number of arithmetic operations required to apply one FFT of order \( 2^k \).

... solving the difference equation we obtain

\[
x_k = \gamma k 2^{k-1} = \frac{\gamma}{2} N \log_2 N = O(N \log N)
\]
Integral equations, convolutions and FFT

We consider the problem of computing potentials from charges

\[ u_j = \sum_k G_{jk} q_k \]

where the matrix \( G \) is related to the Green’s function (or fundamental solution \( \Phi_k \))

¬ Example (PS1)

\[
\int_{\Gamma_k} \Phi_k(x_j - y) \, ds_y \approx \Phi_k(x_j - x_k)|\Gamma_k| \\
= G_{jk}, \quad j \neq k
\]

and \( \psi(x_k) \approx q_k \)

Direct summation requires \( O(N^2) \) operations. Here we present \( O(N \log N) \) FFT-based algorithms for the summation of \( u = Gq \)
Integral equations, convolutions and FFT

- Ring of charges
  translational invariance

\[ G_{jk} = g_{j-k} = g_{k-j} \quad (g_{j-k} = \Phi(|x_j - x_k|)) \]

the potentials are given by \( u_j = \sum_{k=0}^{N} g_{j-k}q_k, \quad j = 0, \ldots, N - 1 \)

\[
\begin{bmatrix}
  u_0 \\
  u_1 \\
  \vdots \\
  u_{N-1}
\end{bmatrix} =
\begin{bmatrix}
  g_0 & g_1 & \cdots & g_{N-1} \\
  g_1 & g_0 & \cdots & g_{N-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{N-1} & g_{N-2} & \cdots & g_0
\end{bmatrix}
\begin{bmatrix}
  q_0 \\
  q_1 \\
  \vdots \\
  q_{N-1}
\end{bmatrix}
\]

\[ g_\ell = g_\ell \mod N, \quad \ell = 0, \ldots, N - 1 \]

therefore \( u = F^{-1}\text{diag}(Fg)(Fq) \)

where \( (F)_{jk} = e^{-2\pi ijk/N} \)
Integral equations, convolutions and FFT

- **Ring of charges**
  - translational invariance

\[
G_{jk} = g_{j-k} = g_{k-j} \quad (g_{j-k} = \Phi(|x_j - x_k|))
\]

the potentials are given by

\[
u_j = \sum_{k=0}^{N} g_{j-k} q_k, \quad j = 0, \ldots, N - 1
\]

This summation is also a cyclic convolution

\[
u = g \ast q \quad (g_{\ell} = g_{\ell \mod N}, \quad \ell = 0, \ldots, N - 1)
\]

Using properties of the DFT it can be easily shown that:

\[
(Fu)_k = (Fg)_k (Fq)_k \quad \text{where} \quad (F)_{jk} = e^{-2\pi ijk/N}
\]
Integral equations, convolutions and FFT

- **Straight rod of charges**

  \[ G_{jk} = g_{j-k} = g_{k-j} \quad (g_{j-k} = \Phi(|x_j - x_k|)) \quad j, k = 0, \ldots, N - 1 \]

  \[
  \begin{bmatrix}
  u_0 \\
  u_1 \\
  \vdots \\
  u_{N-1}
  \end{bmatrix}
  =
  \begin{bmatrix}
  g_0 & g_1 & \cdots & g_{N-1} \\
  g_1 & g_0 & \cdots & g_{N-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{N-1} & g_{N-2} & \cdots & g_0
  \end{bmatrix}
  \begin{bmatrix}
  q_0 \\
  q_1 \\
  \vdots \\
  q_{N-1}
  \end{bmatrix}
  \]

  Toeplitz matrix
\( \tilde{u}_j = \sum_{k=0}^{2N-1} \tilde{g}_{j-k} \tilde{q}_k, \quad j = 0, \ldots, 2N - 1 \)  

cyclic convolution

where \( \tilde{g}_\ell = \tilde{g}_\ell \mod 2N, \quad \ell = 0, \ldots, 2N - 1 \)
### Straight rod of charges

<table>
<thead>
<tr>
<th>$u_0$</th>
<th>$u_1$</th>
<th>\vdots</th>
<th>$u_{N-1}$</th>
<th>$u_N$</th>
<th>$u_{N+1}$</th>
<th>$u_{2N-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_0$</td>
<td>$g_1$</td>
<td>\vdots</td>
<td>$g_{N-1}$</td>
<td>0</td>
<td>$g_{N-1}$</td>
<td>\vdots</td>
</tr>
<tr>
<td>$g_1$</td>
<td>$g_0$</td>
<td>\vdots</td>
<td>$g_{N-2}$</td>
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<td>0</td>
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</tr>
<tr>
<td>$g_{N-1}$</td>
<td>$g_{N-2}$</td>
<td>\vdots</td>
<td>$g_0$</td>
<td>$g_1$</td>
<td>$g_2$</td>
<td>\vdots</td>
</tr>
<tr>
<td>$0$</td>
<td>$g_{N-1}$</td>
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<td>$g_1$</td>
<td>$g_0$</td>
<td>$g_1$</td>
<td>\vdots</td>
</tr>
<tr>
<td>$g_{N-1}$</td>
<td>0</td>
<td>\vdots</td>
<td>$g_2$</td>
<td>$g_1$</td>
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<td>\vdots</td>
<td>$g_0$</td>
<td>$g_1$</td>
<td>$g_2$</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

The circulant matrix generated by $\tilde{g}$

\[
\tilde{u} = F^{-1} \text{diag}(F\tilde{g})(F\tilde{q})
\]

where \[
(F)_{jk} = e^{-2\pi ij k / (2N)}, \quad j, k = 0, \ldots, 2N - 1
\]
Integral equations, convolutions and FFT

Square plate of charges

\[ q_{0,0} \quad q_{0,1} \quad \ldots \quad q_{0,N-2} \quad q_{0,N-1} \]

\[ \bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet \]

\[ \bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet \]

\[ \vdots \quad \vdots \quad \ldots \quad \vdots \quad \vdots \]

\[ q_{N-1,0} \quad q_{N-1,1} \quad q_{N-1,N-2} \quad q_{N-1,N-1} \]

The potentials are given by

\[ u_{j_1,j_2} = \sum_{k_1,k_2}^{N-1} g_{j_1-k_1,j_2-k_2} q_{k_1,k_2}, \quad j_1, j_2 = 0, \ldots, N-1 \]

where \( g_{j_1,j_2} = g_{-j_1,j_2} = g_{-j_1,-j_2} = g_{j_1,-j_2} \) (translational invariance)

For example:

\[ g_{j_1-k_1,j_2-k_2} = \Phi \left( \sqrt{(x_{j_1} - x_{k_1})^2 + (y_{j_2} - y_{k_2})^2} \right) \]
The operation can be turned into a cyclic convolution:

1. Zeropad $q$ into a $2N - 2$ by $2N - 2$ array $\tilde{q}$.

2. Mirror extend $g$ into a $2N - 2$ by $2N - 2$ array $\tilde{g}$ via

\[
\tilde{g} = \begin{pmatrix}
g_{0,0} & g_{0,1} & \cdots & g_{0,N-1} & g_{0,N-2} & g_{0,N-3} & \cdots & g_{0,1} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,N-1} & g_{1,N-2} & g_{1,N-3} & \cdots & g_{1,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g_{N-1,0} & g_{N-1,1} & \cdots & g_{N-1,N-1} & g_{N-1,N-2} & g_{N-1,N-3} & \cdots & g_{N-1,1} \\
g_{N-2,0} & g_{N-2,1} & \cdots & g_{N-2,N-1} & g_{N-2,N-2} & g_{N-2,N-3} & \cdots & g_{N-2,1} \\
g_{N-3,0} & g_{N-3,1} & \cdots & g_{N-3,N-1} & g_{N-3,N-2} & g_{N-3,N-3} & \cdots & g_{N-3,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g_{1,0} & g_{1,1} & \cdots & g_{1,N-1} & g_{1,N-2} & g_{1,N-3} & \cdots & g_{1,1}
\end{pmatrix}
\]
Integral equations, convolutions and FFT

3. Compute the cyclic convolution:

\[ u_{j_1,j_2} = \sum_{k_1,k_2=0}^{2N-3} \tilde{g}(j_1-k_1) \mod (2N-2), (j_2-k_2) \mod (2N-2) \tilde{q}_{k_1,k_2}, \]

using 2D FFT.

4. Restrict the indices of the result to \(0 \leq j_1, j_2 \leq N - 1\).
Consider the integral equation:

\[ \psi(x) = S[\sigma](x) := \int_{\Gamma} \log|\mathbf{x} - \mathbf{y}| \sigma(y) \, ds_y, \quad x \in \Gamma \]

**x_j**: midpoint of panel \( \Gamma_j \)

Discretized integral operator:

\[ \psi(x_i) \approx (Pq)_i \]

where

\[ P_{ij} = \int_{\Gamma_j} \log|\mathbf{x}_i - \mathbf{y}| \, ds_y \quad \text{and} \quad q_j \approx \sigma(\mathbf{x}_j) \]

Fig. 1. Piecewise-constant collocation discretization of two conductors. Conductor surfaces are discretized into panels which support a constant charge density.

Precorrected-FFT method for BIEs

Idea:

1) **Projection**: Project the panel charges onto a uniform grid of points charges (equivalent charges)

2) **FFT**: Compute grid potentials from grid charges using the FFT.

3) **Interpolation**: Interpolate the grid potentials onto the panels.

4) **Near interactions**: Directly compute the nearby interactions.
Precorrected-FFT method for BIEs

Idea:

1) Projection

$k$-th cell

$\hat{x}_i :$ grid points ($\# P^2$)

$x^t_j :$ test points ($\# N_c$)

$x_\ell :$ center of the panel $\Gamma_\ell$
1) Projection

Potential produced by the panel charges $q_\ell$ has to equal the potential produced by grid charges at the test points:

$$\sum_\ell q_\ell \int_{\Gamma_\ell} \log |x^t_i - y| \, ds_y := P^{qt} q(k)$$

potential produced by panel charges in $k$-th cell evaluated at test points

panel charges contained in $k$-th cell

$$\sum_j \hat{q}_j \log |\hat{x}^t_i - \hat{x}_j| := P^{gt} \hat{q}(k)$$

potential produced by grid charges in $k$-th cell evaluated at test points

grid charges associated to $k$-th cell
1) Projection

Potential produced by the panel charges $q_\ell$ has to equal the potential produced by grid charges at the test points:

$$\sum_\ell q_\ell \int_{\Gamma_\ell} \log |x_i^t - y| \, ds_y = P^{qt} q(k) = P^{gt} \hat{q}(k) = \sum_j \hat{q}_j \log |\hat{x}_i^t - \hat{x}_j|$$

$P^{qt}$ does not depend on the panels inside the cell: It is the same matrix for all the cells!

This system can be solved for $\hat{q}(k)$:

$$\hat{q}(k) = (P^{gt})^+ P^{qt} q(k)$$

Moore-Penrose inverse $P^{gt} = U \Sigma V^*$ $\implies$ $\hat{q}(k) = \sum_{\sigma_i \neq 0} \frac{(u_i^* q(k)) v_i}{\sigma_i}$
1) Projection

Potential produced by the panel charges \( q_\ell \) has to equal the potential produced by grid charges at the test points:

\[
\sum_\ell q_\ell \int_{\Gamma_\ell} \log |x_i^t - y| \, ds_y = P^{qt} q(k) = P^{gt} \hat{q}(k) = \sum_j \hat{q}_j \log |\hat{x}_i^t - \hat{x}_j|
\]

\( P^{qt} \) does not depend on the panels inside the cell: It is the same matrix for all the cells!

This system can be solved for \( \hat{q}(k) \):

\[
\hat{q}(k) = (P^{gt})^+ P^{qt} q(k)
\]

We thus define: \( W(k, j) = (P^{gt})^+ P^{qt,j} \)

contribution of the \( j \)-th panel in cell \( k \) to \( \hat{q}(k) \)
2) Compute grid potential

After projection we have grid charges $\hat{q}$. Using the grid charges $\hat{q}$ we can compute grid potentials:

$$\hat{\psi}(i, j) = H\hat{q} = \sum_{i', j'} h(i - i', j - j')\hat{q}(i', j')$$

where

$$h(i - i', j - j') = \begin{cases} \log|x_{i,j} - x_{i',j'}| & \text{if } i \neq i' \text{ or } j \neq j', \\ 0 & \text{otherwise} \end{cases}$$

The grid potentials $\hat{\psi}(i, j)$ can be efficiently evaluated using two two-dimensional FFT's.

In practice the DFT of $H = h(i, j)$ is computed once!
3) Interpolation

We need to obtain the FFT approximated panel potentials $\psi_g$ from the grid potentials $\hat{\psi}$:

**Theorem.** Let $V \in \mathbb{R}^{n \times m}$ be an operator which projects panel charges onto grid charges. Then $V^T$ can be interpreted as an operator that interpolates potentials at grid points onto panel coordinates.

**Note.** In our case the operator that interpolates potentials at grid points in cell $k$ to a charge $j$ also is cell $k$ is not $[W(k, j)]^T$.

Thus, projection, followed by convolution, followed by interpolation gives the grid-charge approximation to the potentials:

$$\psi_g = V^T HWq$$
4) Precorrection

\( P(k, \ell) \): blocks of \( P \) associated with interactions between neighboring cell \( k \) and \( \ell \).

\( V(k) \): matrix formed with columns of \( V(k, j) \)

\( W(k) \): matrix formed with columns of \( W(k, j) \)

\( \psi(k, \ell) \): panel potentials in cell \( k \) due to charges \( q_\ell \) in cell \( \ell \)

\( \psi_g(k, \ell) = V(k)^T H(k, \ell) q(\ell) \): grid approximation to \( \psi(k, \ell) \) (which is inaccurate)

We then define:

\[
\psi(k, \ell) = \psi_g(k, \ell) + \{ P(k, \ell) - V(k)^T H(k, \ell) W(\ell) \} q(\ell)
\]

where

\[
\tilde{P}(k, \ell) = P(k, \ell) - V^T(k) H(k, \ell) W(\ell)
\]

is the precorrected direct interaction operator
Algorithm

**Projection**
set $\hat{q} = 0$
for each cell $k = 1, ..., M$
    for each panel $j$ in cell $k$, $j = 1, ..., n(k)$

$$\hat{q}(k) = \hat{q}(k) + W(k, j)q_j(k)$$

end
end

**Convolution**
compute $\hat{Q} = \text{FFT}(\hat{q})$
compute $\hat{\Psi} = \hat{H}\hat{Q}$
compute $\hat{\psi} = \text{IFFT}(\hat{\Psi})$
Algorithm

**Interpolation**

set $\hat{\psi} = 0$

for each cell $k = 1, ..., M$
  for each panel $j$ in cell $k$, $j = 1, ..., n(k)$
    
    $$\psi_j = \psi_j + [V(k, j)]^T \hat{\psi}_j(k)$$

  end

end

**Nearby interactions**

for each cell $k = 1, ..., M$
  for each cell $\ell$ in $N(k)$
    
    $$\psi(k) = \psi(k) + \tilde{P}(k, \ell)q(\ell)$$

  end

end
Theorem. For a homogeneous distribution of $n$ panels and a given prescribed accuracy, the recorrected-FFT method requires $O(n \log n)$ operations to perform a potential calculation $\psi = Pq$.

Proof.

- Given a computational domain containing $n$ panels
- Assume the space has been divided into an array of $k \times \ell$ cells.
- The number of panels per cell $N_c$ is independent of $n$: $O(k\ell) = O(n)$.
- The grid in each cell is a $p \times p$ array
Performance

Computational cost:

1. Direct interactions: $O(N_c^2 k \ell) = O(k \ell) = O(n)$.

2. Grid projection: $O(np^2) = O(n)$.

3. Grid interpolation: $O(np^2) = O(n)$.

4. Cost of the FFT: $O(p^2 k \ell \log(p^2 k \ell)) = O(n \log n)$

Overall cost: $O(n) + O(n \log n)$