Hierarchical matrices

We consider the linear algebraic point of view behind partitioned low-rank matrices, the fast multipole method, and the calculus of hierarchical matrices.

Let us return to the case of \( x \in A \) and \( y \in B \), with \( A \) and \( B \) well-separated, so we are in effect considering an off-diagonal block of \( G(x, y) \). The variable \( x \) is viewed as a (possibly continuous) row index, while \( y \) is a (possibly continuous) column index. A low-rank expansion of \( G \) is any expression where \( x \) vs. \( y \) appear in separated factors, of the form

\[
G(x, y) = \sum_{n,m} U_n(x) S_{n,m} V_m(y), \quad G = U S V^T.
\]

It is useful to limit the ranges of \( n \) and \( m \) to \( 1 \leq n, m \leq p \), which results in a numerical error:

\[
\left\| G(x, y) - \sum_{n,m}^p U_n(x) S_{n,m} V_m(y) \right\| \leq \epsilon.
\]

The \( \epsilon \)-rank of \( G \) is defined as the smallest \( p \) for which the approximation holds with the spectral norm.
Hierarchical matrices

$$G(x, y) =$$

![Diagram showing hierarchical matrices with arrows labeled A, x, y, and B. The diagram illustrates the relationship between the matrices and variables.]
Hierarchical matrices

We have presented various low rank expansions:

Interpolation: $x \in A, y \in \text{far}(A)$
\[ G(x, y) \approx \sum_m P^A_m(x) G(x^A_m, y). \]

Projection: $y \in B, x \in \text{far}(B)$
\[ G(x, y) \approx \sum_n G(x, y^B_n) P^B_n(y). \]

Multipole expansion: $x \in A, y \in B$, $A$ and $B$ are well separated
\[ G(x, y) \approx \sum_n S_n(x) R_n(y). \]
Low-rank matrices

Let $n, m \in \mathbb{N}$ and $A \in \mathbb{C}^{m \times n}$ be a matrix. The range (or image) of $A$ is defined as

$$\text{Im } A := \{ Ax \in \mathbb{C}^m, x \in \mathbb{C}^n \}.$$

The rank of $A$ is the dimension of its range:

$$\text{rank } A = \dim \text{Im } A.$$

We have the following useful theorem

**Theorem.** Let $m, n, k \in \mathbb{N}$. Then it holds that

- $\text{rank } A \leq \min\{n, m\}$ for all $A \in \mathbb{C}^{m \times n}$;
- $\text{rank}(AB) \leq \min\{\text{rank } A, \text{rank } B\}$ for all $A \in \mathbb{C}^{m \times p}$ and all $B \in \mathbb{C}^{p \times n}$;
- $\text{rank}(A + B) \leq \text{rank } A + \text{rank } B$ for all $A, B \in \mathbb{C}^{m \times n}$.

We denote the set of matrices $A \in \mathbb{C}^{m \times n}$ having at most $k$ linearly independent rows or columns by

$$\mathbb{C}_k^{m \times n} = \{ A \in \mathbb{C}^{m \times n} : \text{rank } A \leq k \}.$$  

(It’s not a linear space)
Low-rank matrices: Representations

Let $A \in \mathbb{C}^{m \times n}$ and note that only $k$ columns of $A$ are sufficient to represent the whole matrix by linear combination. We conclude then that the entry wise representation of $A$ contains redundancies that can be removed.

**Outer-product form.** A matrix $A \in \mathbb{C}^{m \times n}$ can be expressed as

$$A = UV^T = \sum_{i=1}^{k} u_i v_i^T,$$

where $U \in \mathbb{C}^{m \times k}$ and $V \in \mathbb{C}^{n \times k}$. Here $u_i$ and $v_i$, $i = 1, \ldots, k$ denote the columns of $U$ and $V$, respectively.

Instead of storing the $mn$ entries of $A \in \mathbb{C}^{n \times m}$, we can only store the $k$ vectors $u_i, v_i$, $i = 1, \ldots, k$, which require $k(m + n)$ units of storage.

**Orthonormal outer-product form.** A matrix $A \in \mathbb{C}^{m \times n}$ can be expressed as

$$A = U X V^T$$

where $X \in \mathbb{R}^{k \times k}$ and the matrices $U \in \mathbb{C}^{m \times k}$ and $V \in \mathbb{C}^{n \times k}$ have orthonormal columns, i.e., $U^T U = I_k = V^T V$. 
Low-rank matrices

Matrix-vector multiplication. Let \( x \in \mathbb{C}^n \). Then the product \( Ax \in \mathbb{C}^m \) can be obtained following the two-step procedure:
1) Compute \( z = V^T x \in \mathbb{C}^k \); 2) Compute \( y = U z \).
Instead of \( 2mn \) arithmetic operations required by the entry wise multiplication procedure, the outer-product form amounts to only \( 2k(m+n) - (k+m) \) operations.

Low-rank matrix. A matrix \( A \in \mathbb{C}^{m \times n}_k \) is called a low-rank matrix if
\[
k(m+n) < mn.
\]

Sherman-Morrison-Woodbury formula. If \( A, B \in \mathbb{C}^{n \times n} \) are non-singular and \( B \) arises from \( A \) by adding a matrix \( U V^T \in \mathbb{C}^{n \times n}_k \), then provided \( I + V^T A^{-1} U \) is non-singular, also the inverse of \( B \) arises from the inverse of \( A \) by adding a matrix from \( \mathbb{C}^{n \times n}_k \):
\[
B^{-1} = A^{-1} - A^{-1} U (I + V^T A^{-1} U)^{-1} V^T A^{-1}.
\]
Low-rank matrices: Norms

**Frobenius norm.** The Frobenius norm of a matrix $A \in \mathbb{C}^{m \times n}$ is defined as

$$
\|A\|_F = \sqrt{\text{trace}(A^T A)} = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.
$$

Hence, the Frobenius norm of a matrix $A \in \mathbb{C}^{k \times n}$ can be computed using $2k^2(m + n)$ arithmetic operations:

$$
\|UV^T\|_F^2 = \sum_{i,j=1}^{k} (u_i^T u_j)(v_i^T v_j).
$$

**Spectral norm.** The spectral norm of a matrix $A \in \mathbb{C}^{m \times n}$ is defined as $\|A\|_2 = \sqrt{\rho(A^T A)}$ where $\rho(A)$ denotes the spectral radius of $A$. The spectral norm of a matrix $A \in \mathbb{C}^{k \times n}$ can be computed using $O(k^2(n + m))$ arithmetic operations by computing the $k \times k$ matrices $U^T U$ and $V^H V$ and then computing the largest eigenvalue of the product, i.e., $\rho(U^T U V^T V) = \rho(A)$. 
If the matrix $A \in \mathbb{C}^{m \times n}_k$ is given in orthonormal outer-product form

$$A = UXV^T,$$

then:

**Frobenius norm:** $\|UXV^T\|_F = \|X\|_F = \sqrt{\sum_{i,j=1}^{k} |x_{i,j}|^2}$; and

**Spectral norm:** $\|UXV^T\|_2 = \|X\|_2$.

The evaluation of both norms require $O(k^2)$ arithmetic operations.
Adding and Multiplying Low-Rank Matrices

**Multiplication.** Let \( A \in \mathbb{C}^{m \times p} \) and \( B \in \mathbb{C}^{p \times n} \) given in outer-product representation \( A = U_A V_A^T \) and \( B = U_B V_B^T \). The rank of \( AB \) is bounded by \( \min k_A, k_B \). Hence, the outer-product form will be advantageous for the product \( AB \) as well. There are two possibilities for computing \( AB = UV^T \):

1. \( U = U_A (V_A^T U_B) \) and \( V = V_B \) using \( 2k_A k_B (m+p) - k_B (m+k_A) \) operations:
2. \( U = U_A \) and \( V = V_B (U_B^T V_A) \) using \( 2k_A k_B (p+n) - k_A (n+k_B) \) operations.

Depending on the quantities \( k_A, k_B, m, \) and \( n \), either representation should be chosen.

**Addition.** Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{m \times n} \) given in outer-product representation \( A = U_A V_A^T \) and \( B = U_B V_B^T \). Then the sum \( A + B \) will have the following outer-product representation:

\[
A + B = UV^T,
\]

with \( U = [U_A, U_B] \in \mathbb{C}^{m \times k} \) and \( [V_A, V_B] \in \mathbb{C}^{n \times k}, (k = k_A + k_B) \) which ensures that

\[
\text{rank}(A + B) \leq k_A + k_B = k.
\]
Approximation by Low-Rank Matrices

We now state that the closest matrix in $\mathbb{C}^{m \times n}_k$ to a given matrix from $\mathbb{C}^{m \times n}$, $m \geq n$, can be obtained from the singular value decomposition (SVD) $A = U \Sigma V^T$ with $U^T U = I_n = V^T V$ and a diagonal matrix $\Sigma \in \mathbb{R}^{n \times n}$ with entries $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$.

**Theorem.** Let the SVD $A = U \Sigma V^T$ of $A \in \mathbb{C}^{m \times n}$, $m \geq n$, be given. Then for $k \in \mathbb{N}$ satisfying $k \leq n$ it holds that

$$\min_{M \in \mathbb{C}^{m \times n}_k} \|A - M\| = \|A - A_k\| = \|\Sigma - \Sigma_k\|,$$

where $A_k = U \Sigma_k V^H \in \mathbb{C}^{m \times n}_k$ and $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0) \in \mathbb{R}^{n \times n}$.

In spectral norm we have:

$$\|A - A_k\|_2 = \sigma_{k+1},$$

while in Frobenius norm we have:

$$\|A - A_k\|_F^2 = \sum_{l=k+1}^{n} \sigma_l^2.$$
Approximation by Low-Rank Matrices

The computation of the singular value decomposition of a general matrix $A \in \mathbb{C}^{m \times n}$ ($m \geq n$) matrix costs $14mn^2 + 8n^3$ operations (Golub & Van Loan, “Matrix Computations”).

**QR decomposition.** Let $A \in \mathbb{C}^{m \times n}$ then:

1. An orthogonal matrix $Q \in \mathbb{C}^{m \times m}$ and an upper triangular $R \in \mathbb{C}^{m \times n}$ matrix exist with $A = QR$.

2. If $m > n$, $R$ has the block structure $R = [R' \, 0]^T$, where the submatrix $R' \in \mathbb{C}^{n \times n}$ is a square upper triangular matrix. The corresponding block partition $Q = [Q' \, \ast]$ together with $M = QR = Q'R'$ yields the reduced QR decomposition $M = Q'R'$.

The number of operations required to compute the reduced QR decomposition is $4mn^2$. 
Approximation by Low-Rank Matrices

Reduced singular value decomposition Let \( A = UV^T \), \( U \in \mathbb{C}^{m \times k} \) and \( V \in \mathbb{C}^{n \times k} \). Then the reduced singular value decomposition \( A_k \) of \( A \) is obtained as follows:

1. Compute the reduced QR decomposition \( U = Q_U R_U \) with an orthonormal \( Q_U \in \mathbb{C}^{m \times k} \) and an upper triangular matrix \( R_U \in \mathbb{C}^{k \times k} \).

2. Compute the reduced QR decomposition \( V = Q_V R_V \) with an orthonormal \( Q_V \in \mathbb{C}^{n \times k} \) and an upper triangular matrix \( R_V \in \mathbb{C}^{k \times k} \).

3. Compute the singular value decomposition of \( R_U R_V^T = \hat{U} \Sigma \hat{V}^T \) (all matrices in \( \mathbb{C}^{k \times k} \)).

4. Define \( U = Q_U \hat{U} \in \mathbb{C}^{m \times k} \) and \( V = Q_V \hat{V} \in \mathbb{C}^{n \times k} \).

Then \( A_k = U \Sigma V^T \) is the reduced singular value decomposition of \( A \) and its computation requires less than \( 6k^2(n + m) + 4k^3 \) operations.

Truncation. Given a rank-\( k \) matrix \( A \in \mathbb{C}^{m \times n}_k \) with \( k > r \). Then \( 6k^2(n + m) + 22k^3 \) operations are needed to determine the optimal rank-\( r \) matrix \( A_r \in \mathbb{C}^{m \times n}_r \).
Approximation by Low-Rank Matrices

**Formatted addition.** Let $A \in \mathbb{C}^{m \times n}_{k_A}$ and $B \in \mathbb{C}^{m \times n}_{k_B}$. We showed that $C = A + B \in \mathbb{C}^{m \times n}_{k_A+k_B}$. It is reasonable to truncate the sum to a smaller rank $k < k_A + k_B$. Such truncation is called *formatted addition*. Using the reduced singular value decomposition the formatted addition of the matrices $A$ and $B$ costs $O((k_A + k_B)^2(m + n))$ operations.

**Formatted agglomeration.** Let two matrices $A \in \mathbb{C}^{m \times n}_{k_A}$ and $B \in \mathbb{C}^{m \times n}_{k_B}$. As is the case of addition, the matrix $C = [A \ B] \in \mathbb{C}^{m \times n}_{k_A+k_B}$. The formatted agglomeration of the matrices $A$ and $B$ is defined as the formatted addition of the matrices $[A \ 0]$ and $[0 \ B]$. The cost of the formatted agglomeration is less that the cost of the formatted addition because of the zero block structure of $[A \ 0]$ and $[0 \ B]$ and amounts to $O(k^2(n + m) + k^3)$ operations.
Consider matrices $A \in \mathbb{C}^{n \times n}$, where $n$ is a power of two: $n = 2^p$. We define the matrix format $\mathcal{H}_p$ recursively starting with the representation $\mathcal{H}_0$ which corresponds to scalars ($1 \times 1$ matrices). Assuming that the format $\mathcal{H}_{p-1}$ for matrices $\mathbb{C}^{2^{p-1} \times 2^{p-1}}$ is known, then a matrix $A \in \mathbb{C}^{2^p \times 2^p}$ can be represented as the following block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{ij} \in \mathbb{C}^{2^{p-1} \times 2^{p-1}}.$$ 

We then restrict the set of all $A \in \mathbb{C}^{2^p \times 2^p}$ by the conditions

$$A_{11}, A_{22} \in \mathcal{H}_{p-1}, \quad A_{12}, A_{21} \in \mathbb{C}_k^{2^{p-1} \times 2^{p-1}}.$$ 

$p = 0$

\[\begin{array}{c}
\end{array}\]

$p = 1$

\[\begin{array}{c|c|c|c}
\hline & & & \\
\hline & & & \\
\hline & & & \\
\hline & & & \\
\hline
\end{array}\]

$p = 2$

\[\begin{array}{c|c|c|c}
\hline & & & \\
\hline & & & \\
\hline & & & \\
\hline & & & \\
\hline
\end{array}\]

$p = 3$

\[\begin{array}{c|c|c|c}
\hline & & & \\
\hline & & & \\
\hline & & & \\
\hline & & & \\
\hline
\end{array}\]
Model format $\mathcal{H}_p$

**Number of blocks in the $\mathcal{H}_p$ format.** Start with $p = 0$. The $1 \times 1$ matrix contains $N(0) = 1$ block. Now by the recursion we have $N(p) = 2 + 2N(p - 1)$ for $p > 0$. Therefore, we conclude that

$$N(p) = 3n - 2, \quad n = 2^p.$$ 

**Storage cost.** The storage cost required for a matrix $A \in \mathbb{C}_k^{2^p \times 2^p}$ is $k \cdot 2^{p+1}$. Let $S(p)$ be the storage cost of a matrix in $\mathcal{H}_p$. For $p = 0$ we have $S(0) = 1$. The recursion formula shows that

$$S(p) = k \cdot 2^p + 2S(p-1).$$

Therefore, solving the difference equation with initial condition $S(0) = 1$ we obtain $S(p) = (pk + 1)2^p = (k \log_2 n + 1)n$. 

**Mat-vec product.** Let $N(p)$ denotes the cost of a matrix-vector multiplication $Ax$, where $A \in \mathcal{H}_p$. For $p \geq 1$ we write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $A_{ij} \in \mathbb{C}^{2^{p-1} \times 2^{p-1}}$ and $x_j \in \mathbb{C}^{2^{p-1}}$, $i, j = 1, 2$. Therefore the multiplication requires computing the products $y_{11} = A_{11}x_1$, $y_{12} = A_{12}x_2$, $y_{21} = A_{21}x_1$ and $y_{22} = A_{22}x_2$ and the sums $y_{11} + y_{12}$ and $y_{21} + y_{22}$. Since $A_{21}, A_{12} \in \mathbb{C}_k^{2^{p-1} \times 2^{p-1}}$ the cost of each multiplication is $(4k - 1)n/2 - k$, whereas one addition takes $n/2$ operations. Therefore

$$N(p) = 2N(p-1) + 2((4k - 1)n/2 - k + n/2) = 2N(p-1) + 2k(2^{p+1} - 1).$$

Solving the difference equation we obtain:

$$N(p) = 4kn \log_2 n + 2k - 2kn.$$
Matrix addition. We distinguish three types of additions:

1. $A + B \in \mathbb{C}_k^{2^p \times 2^p}$ (in the sense of formatted addition) where $A, B \in \mathbb{C}_k^{2^p \times 2^p}$ and cost $N_{R+R}(p)$.

2. $A + B \in \mathcal{H}_p$ where $A, B \in \mathcal{H}_p$ and cost $N_{H+H}(p)$.

3. $A + B \in \mathcal{H}_p$ where $A \in \mathcal{H}_p$ and $B \in \mathbb{C}_k^{2^p \times 2^p}$ with cost $N_{H+R}(p)$.
Algebraic operations in $\mathcal{H}_p$

1. $N_{R+R}$ is the cost of the formatted addition of two matrices in $\mathbb{C}_k^{2^p \times 2^p}$: 
   $N_{R+R}(p) = \mathcal{O}(k^2 2^p) + \mathcal{O}(k^3)$.

2. Letting $\mathcal{R}_p = \mathbb{C}_k^{2^p \times 2^p}$ we have that the sum has the structure
   \[
   \begin{bmatrix}
   \mathcal{H}_{p-1} & \mathcal{R}_{p-1} \\
   \mathcal{R}_{p-1} & \mathcal{H}_{p-1}
   \end{bmatrix}
   + \begin{bmatrix}
   \mathcal{H}_{p-1} & \mathcal{R}_{p-1} \\
   \mathcal{R}_{p-1} & \mathcal{H}_{p-1}
   \end{bmatrix}
   = \begin{bmatrix}
   \mathcal{H}_{p-1} + \mathcal{H}_{p-1} & \mathcal{R}_{p-1} + \mathcal{R}_{p-1} \\
   \mathcal{R}_{p-1} + \mathcal{R}_{p-1} & \mathcal{H}_{p-1} + \mathcal{H}_{p-1}
   \end{bmatrix}.
   \]
   Therefore
   \[
   N_{H+H}(p) = 2N_{H+H}(p - 1) + Ak^2 2^p + Bk^3 \quad (A, B \text{ constants}).
   \]
   Thus $N_{H+H} = (1 + Bk^3)2^p + Ak^2 p 2^p - Bk^3 = (1 + Bk^3)n + Ak^2 n \log_2 n - Bk^3$.

3. The sum has the structure
   \[
   \begin{bmatrix}
   \mathcal{H}_{p-1} & \mathcal{R}_{p-1} \\
   \mathcal{R}_{p-1} & \mathcal{H}_{p-1}
   \end{bmatrix}
   + \begin{bmatrix}
   \mathcal{R}_{p-1} & \mathcal{R}_{p-1} \\
   \mathcal{R}_{p-1} & \mathcal{R}_{p-1}
   \end{bmatrix}
   = \begin{bmatrix}
   \mathcal{H}_{p-1} + \mathcal{R}_{p-1} & \mathcal{R}_{p-1} + \mathcal{R}_{p-1} \\
   \mathcal{R}_{p-1} + \mathcal{R}_{p-1} & \mathcal{H}_{p-1} + \mathcal{R}_{p-1}
   \end{bmatrix}.
   \]
   Thus $N_{H+R} = (1 + Bk^3)2^p + Ak^2 p 2^p - Bk^3 = (1 + Bk^3)n + Ak^2 n \log_2 n - Bk^3$. 
Matrix-matrix multiplication. Let $\mathbb{C}^{2p \times 2p} = \mathcal{R}_p$. We distinguish four cases

1. $A \cdot B \in \mathcal{R}_p$ for $A, B \in \mathcal{R}_p$ (cost $N_{R \cdot R}(p)$).
2. $A \cdot B \in \mathcal{R}_p$ for $A \in \mathcal{R}_p$ and $B \in \mathcal{H}_p$ (cost $N_{R \cdot H}(p)$).
3. $A \cdot B \in \mathcal{R}_p$ for $A \in \mathcal{H}_p$ and $B \in \mathcal{R}_p$ (cost $N_{H \cdot R}(p)$).
4. $A \cdot B \in \mathcal{H}_p$ for $A, B \in \mathcal{H}_p$ (cost $N_{H \cdot H}(p)$).
Algebraic operations in $\mathcal{H}_p$

1. $N_{R\cdot R} = O(k^2 2^p) = O(k^2 n)$ (to produce the outer product form of the multiplication).

2. Let $A \in \mathcal{H}_p$ and $B \in \mathcal{R}_p$. We have $AUV^T = (AU)V^T$ where $AU \in \mathcal{R}_p$. $AU$ is equivalent to perform $k$ matrix-vector products which costs $O(kn \log_2 n)$ operations.

3. Let $A \in \mathcal{R}_p$ and $B \in \mathcal{H}_p$. Then $AB = UV^T B = U(V^T B) = U(B^T V)^T$. $B^T V \in \mathcal{R}_p$ requires $O(kn \log_2 n)$ operations.

4. Let $A \in \mathcal{H}_p$ and $B \in \mathcal{H}_p$. The product has the form

$$\begin{bmatrix} \mathcal{H}_{p-1} & \mathcal{R}_{p-1} \\ \mathcal{R}_{p-1} & \mathcal{H}_{p-1} \end{bmatrix} \cdot \begin{bmatrix} \mathcal{H}_{p-1} & \mathcal{R}_{p-1} \\ \mathcal{R}_{p-1} & \mathcal{H}_{p-1} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{p-1} \cdot \mathcal{H}_{p-1} + \mathcal{R}_{p-1} \cdot \mathcal{R}_{p-1} & \mathcal{H}_{p-1} \cdot \mathcal{R}_{p-1} + \mathcal{R}_{p-1} \cdot \mathcal{H}_{p-1} \\ \mathcal{R}_{p-1} \cdot \mathcal{H}_{p-1} + \mathcal{H}_{p-1} \cdot \mathcal{R}_{p-1} & \mathcal{R}_{p-1} \cdot \mathcal{R}_{p-1} + \mathcal{H}_{p-1} \cdot \mathcal{H}_{p-1} \end{bmatrix}$$

The cost is then given by

$$N_{H\cdot H}(p) = 2N_{H\cdot H}(p-1) + 2N_{R\cdot R}(p-1) + N_{H\cdot R}(p-1) + N_{R\cdot H}(p-1) + 2N_{H+R}(p) + N_{R+R}(p)$$

Thus $N_{H\cdot H}(p) = 2N_{H\cdot H}(p-1) + O(n \log_2 n) + O(n)$, which yields

$$N_{H\cdot H}(p) = O(n \log^2_2 n) + O(n \log_2 n) + O(n)$$

operations.
**Algebraic operations in \( \mathcal{H}_p \)**

**Matrix inversion.** We want to approximate the inverse \( A^{-1} \) of a matrix \( A \in \mathcal{H}_p \). We define the inversion recursively. For \( p = 0 \) the inverse is exact. For \( p \geq 1 \) the exact inverse of \( A \) has the block structure:

\[
A^{-1} = \begin{bmatrix}
A_{11}^{-1} + A_{11}^{-1} A_{12} S^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} S^{-1} \\
-S^{-1} A_{21} A_{11}^{-1} & S^{-1}
\end{bmatrix}
\]

involving the Schur complement \( S = A_{22} - A_{21} A_{11}^{-1} A_{12} \). The cost of computing the approximate inverse \( A^{-1} \) amounts to

\[
N_{\text{inv}}(p) = 2N_{\text{inv}}(p-1) + 2N_{R \cdot H}(p-1) + 2N_{H \cdot R}(p-1) + 2N_{H+R}(p-1) + 2N_{R \cdot R}(p-1)
\]

Therefore \( N_{\text{inv}}(p) = O(n \log_2 n) + O(n \log_2 n) \).
Example: 1D Green’s matrix

We already justified the separability of the off-diagonal blocks of the Laplace and Helmholtz Green’s functions in 2D. We now consider, from an algebraic viewpoint, the separability of the Green’s matrix corresponding to the discretization of a 1D Dirichlet boundary value using a 3-point finite difference stencil.

The finite-difference discretization of a 1D boundary value problem leads to the tridiagonal matrix:

\[
T = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\gamma_1 & \alpha_2 & \ddots \\
& \ddots & \ddots & \beta_{n-1} \\
& & \gamma_{n-1} & \alpha_n
\end{bmatrix}.
\]

The inverse \( G = T^{-1} \) corresponds to the approximation of the Green’s function of the problem.
Theorem. Assume $T$ is tridiagonal and invertible. The $G = T^{-1}$ has rank-1 off-diagonal blocks.

Proof.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
-1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Corollary. Let $T$ be banded with bandwidth $2p + 1$, then $T^{-1}$ has rank-$p$ off-diagonal blocks.