# RESEARCH STATEMENT

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I am an algebraic geometer with broad interests, including complex dynamics, several complex variables, combinatorics and number theory. My research focuses on topological and numerical invariants of moduli spaces of curves and surfaces, properties of rationally connected varieties, the cohomology of homogeneous varieties and Gromov-Witten theory. Below I will detail my contributions to each of these subjects and describe future research proposals.

1.1. **Characteristic numbers.** Objects of algebraic geometry with fixed discrete invariants (e.g., curves of genus $g$ and degree $d$ in $\mathbb{P}^n$) are often parameterized by finite dimensional algebraic parameter spaces. Imposing geometric constraints on the objects (e.g., requiring the curves to intersect a fixed linear space) defines subvarieties of these parameter spaces. When such subvarieties have zero-dimensional intersection, the number of points in their intersection is an important geometric invariant.

A classical problem dating back to the nineteenth century, known as the *characteristic number problem*, is the problem of finding the number of varieties in projective space incident to the 'appropriate number' of general linear spaces. In the last decade, using ideas from string theory (see [KM]), mathematicians have determined many characteristic numbers of curves. In comparison, the characteristic numbers of higher-dimensional varieties are harder to compute, hence the progress has been more modest.

In my thesis, I studied the degenerations of higher-dimensional varieties of minimal degree in projective space. The study of degenerations provides invaluable insight into the geometry of a variety. It is often easier to deduce geometric properties of a variety by studying the specializations of that variety. Moreover, understanding degenerations is an important component of moduli theory. In my thesis, I gave a detailed description of degenerations of scrolls in projective space.

Degeneration techniques were successfully applied by Caporaso and Harris ([CH]) to determine the degrees of Severi varieties of plane curves and by Vakil ([V1]) to determine the characteristic numbers of rational and elliptic curves in projective space. In [C1], using my analysis of degenerations of scrolls, I obtained an algorithm for computing characteristic numbers of rational normal surface scrolls. A typical result that can be deduced using the algorithm is the following:

**Proposition 1.1.** *The number of rational surface scrolls of degree $n$ in $\mathbb{P}^{n+1}$ containing $n+5$ general points and intersecting a general $(n-3)$-dimensional linear space is $(n-1)(n-2)$.*

The enumerative geometry of rational scrolls is important from many perspectives. Rational scrolls, together with cones over the Veronese surface in $\mathbb{P}^5$, are the non-degenerate projective varieties of minimal degree. They play a central role in many constructions of Castelnuovo Theory and moduli theory. Their enumerative geometry is especially attractive for the connection to Gromov-Witten theory. There is a natural correspondence between rational curves in the Grassmannian and rational scrolls. Using this correspondence, I obtained an efficient algorithm to compute the Gromov-Witten invariants of Grassmannians when enough of the conditions are intersection conditions with small-dimensional linear spaces.

In [C2], I investigated the degenerations of Del Pezzo surfaces embedded in projective space by their anti-canonical bundle. The Del Pezzo surfaces are abstractly $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of the projective plane $D_n$ in $9-n$ general points. When they are embedded by their anti-canonical bundle, they are varieties of one more than the minimal possible degree.

The enumerative geometry of Del Pezzo surfaces is more delicate than the enumerative geometry of scrolls. Scrolls degenerate into unions of scrolls, hence their enumerative geometry can be determined recursively. Del Pezzo surfaces exhibit many different degenerations. For example, $D_n$ can degenerate into a union of two scrolls, a union of a Veronese surface and a scroll, $D_{n-1}$ union a plane, an elliptic cone union a rational cone, the projection of a rational scroll or a Veronese surface with a double line. In [C2], these degenerations were constructed using the fact that $D_n$ is swept out by one-parameter families of conics. We reinterpret $D_n$ as a curve in the Hilbert scheme of conics in $\mathbb{P}^n$. Using the deformation theory of curves on the Hilbert scheme, we exhibit families realizing the degenerations.

Given the diversity of different degenerations, one cannot hope for a recursive formula for the characteristic numbers of $D_n$. A theorem of Del Pezzo and Nagata [Na] on the classification of surfaces of degree $n$ in $\mathbb{P}^n$ places many constraints on the possible limits of Del Pezzo surfaces. When $n = 3, 4$ or 5, the limits can be analyzed explicitly. This was carried out in [C2] to obtain new enumerative results. Our research also demonstrates that there is a real qualitative difference in difficulty between degenerations of curves and degenerations of higher-dimensional varieties. This is consistent with the results of Vakil in [V2].

My thesis work concentrated on counting balanced rational scrolls. A rational scroll abstractly is the projectivization of a vector bundle on $\mathbb{P}^1$. By Grothendieck's theorem such a vector bundle is the direct sum of line bundles. The scroll is balanced if the degrees of any two summands differ by at most one. The enumerative geometry of unbalanced scrolls is equally interesting. In [C4], I developed a method for computing the characteristic numbers of rational scrolls of any dimension and any splitting type. This work was inspired by a question of I. Vainsencher (see [VX] and [LVX]). The solution requires us to translate the problem to a problem of counting rational curves with a given splitting type with respect to the tautological bundle of the Grassmannian.

More generally, let $X$ be a smooth, projective variety. Let $E$ be a rank $k$ vector bundle on $X$. Let $\beta$ be a curve class on $X$. For any map $f : \mathbb{P}^1 \to X$ in the Kontsevich space of $m$-pointed genus-zero maps $M_{0,m}(X, \beta)$ we say that $f$ has splitting type $(r_1, \ldots, r_k)$ for the vector bundle $E$ if the degrees of the line bundles in the Grothendieck decomposition of $f^*E$ on $\mathbb{P}^1$ are $(r_1, \dots, r_k)$. Let $i_1^{b_1}, \dots, i_j^{b_j}$ denote the splitting type

$$r_1 = \cdots = r_{b_1} = i_1, \quad r_{b_1+1} = \cdots = r_{b_1+b_2} = i_2, \quad \dots, \quad r_{k-b_j+1} = \cdots = r_k = i_j.$$

Set $a_i = \sum_{h=1}^{i} b_h$. The partial flag variety $F(a_1, \dots, a_j; n)$ admits projections

$$\pi_h : F(a_1, \dots, a_j; n) \to G(a_h; n)$$

to Grassmannians for every $1 \leq h \leq j$. The class of a curve in the flag variety is determined by the Plücker degrees of its $j$ projections. We will denote the curve classes by $(d_1, \dots, d_j)$ where $d_i$ is the Plücker degree of the curve under the $i$-th projection. Let $\gamma_1, \dots, \gamma_m$ be classes of Schubert cycles in $G(k, n)$ whose codimensions sum to the dimension of the locus of maps of type $i_1^{b_1}, \dots, i_j^{b_j}$. Let $\pi_j^* \gamma_i$ denote the pull-back of the Schubert cycles to the flag variety by the projection $\pi_j$. With this notation we can phrase the main observation as follows.

**Proposition 1.2.** *Let $E$ be the tautological $k$-plane bundle on the Grassmannian $G(k, n)$. The number of maps of type $(i_1^{b_1}, \dots, i_j^{b_j})$ for $E$ intersecting general representatives of the Poincaré duals of the Schubert classes $\gamma_1, \dots, \gamma_m$ is equal to the Gromov-Witten invariant of $F(a_1, \dots, a_j; n)$ associated to the curve class $(b_1 i_1, b_1 i_1 + b_2 i_2, \dots, \sum_h b_h i_h)$ and Schubert classes $\pi_j^* \gamma_1, \dots, \pi_j^* \gamma_m$.*

This proposition can be interpreted as a generalization of the "kernel-span technique" (see [BKT]). We recall that Buch defines the kernel of a curve as the intersection of all the $a$-planes parameterized by the curve. Unfortunately, the kernel of a rational curve $C$ in the Grassmannian $G(a, n)$ is almost always empty unless the curve has very small degree or is very special. We can replace the kernel by another natural invariant: the sequence of minimal subscrolls associated to the curve. In case the minimal subscrolls are the vertices of a cone we recover the kernel. The advantage of using minimal subscrolls is that every irreducible rational curve in the Grassmannian has an associated sequence of minimal subscrolls.

Proposition 1.2 explains the "unexpected" vanishing of Gromov-Witten invariants of partial flag varieties. In order for a Gromov-Witten invariant not to vanish it is not enough for the codimensions of the classes $\gamma_i$ to add up to the dimension of the Kontsevich moduli space. The conditions must allow the existence of the appropriate collection of minimal subscrolls. For example, one may deduce the following corollary.

**Corollary 1.3.** *Let $m \geq 3$, $2a \leq n$ and $d + a \leq n$. The $m$-pointed degree $d$ genus-zero Gromov-Witten invariants of $G(a, n)$ vanish unless*

$$d + \frac{m-3}{d} \leq (m-2)a.$$

This was observed for $m = 3$ in [BKT]. Another corollary is the solution of the characteristic number problem.

**Corollary 1.4.** *Let $(r_1, \dots, r_k)$ be equal to the splitting type $(i_1^{b_1}, \dots, i_j^{b_j})$. The characteristic numbers of scrolls $S_{r_1, \dots, r_k}$ in $\mathbb{P}^N$ having $m$ fibers satisfying the Schubert conditions $\gamma_1, \dots, \gamma_m$ are equal to the Gromov-Witten invariants of $F(a_1, \dots, a_j; N+1)$ for* *the curve class* $(i_1 b_1, \ldots, \sum_{h=1}^{j} i_h b_h)$ *associated to the cohomology classes* $\pi_j^* \gamma_1, \ldots, \pi_j^* \gamma_m$, *except when the scroll is a balanced scroll of degree 1 or 2.*

More generally, this approach gives a way of defining and calculating the invariants of jumping curves for a vector bundle on an arbitrary projective variety (see [C4]). These jumping numbers are interesting because of their relation to the famous Hartshorne conjecture.

**Problem 1.5.** *Are there any indecomposable rank 2 vector bundles on* $\mathbb{P}^n$ *for* $n \geq 5$*?*

The non-existence of such bundles is Hartshorne's conjecture and is related to the question of whether there are any codimension two smooth subvarieties in $\mathbb{P}^n$ for $n \geq 6$ that are not complete intersections (see [Hu]). There are indecomposable rank 2 bundles on $\mathbb{P}^4$. The Horrocks-Mumford bundle is the most famous example. In fact, up to standard constructions such as pulling-back by branched covers and tensoring by line bundles, the Horrocks-Mumford bundle is the only known example of an indecomposable rank 2 bundle on $\mathbb{P}^4$. Consequently, it is interesting to count the jumping curves of the Mumford-Horrocks bundle. I determined these invariants in [C4].

Despite the progress many aspects of the characteristic number problem remain open. A complete answer to the characteristic number problem is impossible. However, the problem may be approached using similar techniques for other special varieties. For instance, the following problems deserve further investigation.

**Problem 1.6.** *Study the enumerative geometry of complete intersection varieties using degenerations or Gromov-Witten theory.*

**Problem 1.7.** *Study the degenerations of K3 surfaces. Use this study to solve the characteristic number problem at least under certain restrictions on the polarization of the K3.*

**Problem 1.8.** *Study the degenerations of canonically embedded curves in projective space. Using this study solve the characteristic number problem for canonical curves.*

These problems are interesting because of their connections to other parts of algebraic geometry and mathematics. I believe they would also make good graduate student projects. They are concrete, yet they require a broad set of techniques from many parts of algebraic geometry to solve. Currently, I have two second year graduate students: Craig Desjardin and Brian Lehmann. If they show interest, I will get them to think about some of these problems.

1.2. **The cones of ample and effective divisors on Kontsevich moduli spaces.** The cones spanned by ample and effective divisors are among the most important invariants of a variety. These cones dictate the birational geometry of a variety. Their study for the moduli spaces of curves has led to fundamental insights to the theory of curves, enabling Eisenbud, Harris and Mumford to prove that the moduli space of curves is of general type when $g > 23$ ([HM], [H], [EH]).

In joint work with Joe Harris and Jason Starr [CHS1], we determined the ample cones of the closely related Kontsevich moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$. In [Pa], R. Pandharipande gives generators for the Picard group of the Kontsevich space:

(1) the class $\mathcal{H}$ of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$ (provided $r > 1$ and $d > 0$),
(2) the class $\mathcal{L}_i$ of the pull-back $\mathrm{ev}_i^*(\mathcal{O}_{\mathbb{P}^r}(1))$, for $1 \leq i \leq n$, associated to the $i^{\text{th}}$ evaluation morphism, $\mathrm{ev}_i(C, (p_1, \ldots, p_n), f) := f(p_i)$,
(3) and the classes $\Delta_{(A,d_A),(B,d_B)}$ of the boundary divisors consisting of maps with reducible domains. Here $A \sqcup B$ is any ordered partition of the marked points, and $d_A$ and $d_B$ are non-negative integers satisfying $d = d_A + d_B$. If $d_A = 0$ (respectively, if $d_B = 0$), we demand $\#A \geq 2$ (resp. $\#B \geq 2$).

For $d \geq 2$, there is another NEF and base-point-free divisor class $\mathcal{T}$, the *tangency divisor*: Fixing a hyperplane $\Pi \subset \mathbb{P}^r$, $\mathcal{T}$ is the class of the divisor parameterizing stable maps $(C, (p_1, \ldots, p_n), f)$ for which $f^{-1}(\Pi)$ is not $d$ reduced, smooth points of $C$. We recall that NEF divisors are the divisors that lie in the closure of the open cone of ample divisors. Our main theorem identifies the ample cone in terms of these divisor classes.

**Theorem 1.9.** *Let $r$ and $d$ be positive integers, $n$ a nonnegative integer such that $n+d \geq 3$. There is an injective linear map,*

$$v : Pic(\overline{\mathcal{M}}_{0,n+d})_{\mathbb{Q}}^{\mathfrak{S}_d} \to Pic(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d))_{\mathbb{Q}}.$$

*The NEF cone of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$, respectively, the base-point-free cone, is the product of the cone generated by $\mathcal{H}, \mathcal{T}, \mathcal{L}_1, \ldots, \mathcal{L}_n$ and the image under $v$ of the NEF cone of $\overline{\mathcal{M}}_{0,n+d}//\mathfrak{S}_d$, respectively, the base-point-free cone.*

The action of $\mathfrak{S}_d$ on the moduli space of genus-zero curves with $n + d$ marked points $\overline{\mathcal{M}}_{0,n+d}$ permutes the last $d$ marked points. The map $v$ generates NEF and base-point-free divisors on the Kontsevich space from NEF and base-point-free divisors on $\overline{\mathcal{M}}_{0,n+d}//\mathfrak{S}_d$. In particular, it generates contractions of the Kontsevich space from contractions of $\overline{\mathcal{M}}_{0,n+d}//\mathfrak{S}_d$. A useful corollary is the following theorem which asserts that the boundary of the Kontsevich moduli space may be contracted.

**Theorem 1.10.** *For every integer $r \geq 1$ and $d \geq 2$, there is a contraction,*

$$cont : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d) \to Y,$$

*restricting to an open immersion on the interior $\mathcal{M}_{0,0}(\mathbb{P}^r, d)$ and whose restriction to the boundary divisor $\Delta_{k,d-k} \cong \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, k) \times_{\mathbb{P}^r} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, d-k)$ factors through the projection to $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, d-k)$ for each $1 \leq k \leq \lfloor d/2 \rfloor$. The following divisor is the pull-back of an ample divisor on $Y$,*

$$D_{r,d} = \mathcal{T} + \sum_{k=2}^{\lfloor d/2 \rfloor} k(k-1)\Delta_{k,d-k}.$$

The contraction in Theorem 1.10 has been independently constructed by A. Parker [Par] and Anca Mustaţă and Andrei Mustaţă (see [MM]).

Our study of the ample cone has applications to the theory of rational curves on Fano manifolds. Using our results, Johan de Jong and Jason Starr, have proved the existence (under mild assumptions) of rational surfaces through a general point on Fano manifolds $X$ with pseudo-index at least three and NEF second chern character (i.e., $c_1^2(X) \geq 2c_2(X)$ when restricted to every effective surface) [dJS].

In another joint project with Joe Harris and Jason Starr, we studied the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, d)$ when $r \geq d$. It is especially interesting to understand the case when $n = d$ in view of the following proposition.

**Proposition 1.11.** *When one expresses the effective cone in terms of the standard generators of the Picard group, $H$ and $\Delta_{i,d-i}$ for $1 \leq i \leq d/2$, the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, d)$, is equal to the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ when $n \geq d$.*

In [CHS2], we obtain the following complete description of the effective cone of the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$.

**Theorem 1.12.** *The effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is generated by positive linear combinations of the boundary divisors and the divisor of degenerate curves.*

If there is an irreducible curve whose deformations pass through a general point of a variety, then that curve must have non-negative intersection number with every effective divisor. Such curves are called moving curves. Each moving curve determines an inequality on the effective cone. We prove the theorem by exhibiting enough moving curves.

Our construction of moving curves emphasizes the relation between the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ and the celebrated Harbourne-Hirschowitz conjecture (see [Har], [Hir]). The Harbourne-Hirschowitz conjecture asserts that a linear system on a general blow-up of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ is non-special if and only if it does not contain any multiple $(-1)$-curves in its base locus. In fact, we prove the theorem by resolving special cases of the Harbourne-Hirschowitz conjecture:

**Proposition 1.13.** *Let $k, j$ and $d$ be positive integers subject to the condition that $2k \leq d$. There exists an integer $n(k,d)$ depending only on $k$ and $d$ such that the linear system*

$$L'(j) = d\ F_1 + \left(\frac{jk(k+1)}{2} - 1\right)\ F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} k\ E_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

*on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $j(d+1) + n(k,d)\frac{(k-1)(k+2)}{2}$ general points is non-special for every $j > 0$. The integer $n(k,d)$ may be taken to be*

$$n(k,d) = \lceil 2(d+1)/k \rceil.$$

These results extend to moduli spaces of stable maps to other targets. In joint work with Jason Starr [CS], we determined the ample cone when the target is a homogeneous variety and we described the stable effective cone when the target is a Grassmannian.

Our work on the ample and effective cone of the Kontsevich moduli spaces suggests approaches for answering corresponding questions on the moduli space of curves. Recently, G. Farkas and M. Popa gave a counterexample to the slope conjecture (see [FaP]) of Harris and Morrison (see [HMo]). This conjecture asserted that the slope of an effective divisor on $\overline{M}_g$ is bounded by the slope of the Brill-Noether divisor(s). Unfortunately, no alternative descriptions of the effective cone have emerged since the counterexample.

Our work on the effective cone suggests using moving curves for obtaining lower bounds on the slope of the moduli space of curves. For instance, consider one-parameter families of canonical curves of genus $g$ in $\mathbb{P}^{g-1}$ defined by requiring the curves to intersect general linear spaces. These one-parameter families are especially interesting when the maximum possible number of linear spaces are points. The analogy with the case of rational curves and the fact that in small genus these families give sharp bounds leads us to believe that this sequence of moving curves may improve the bounds on the slope. It may even give a non-zero bound as the genus tends to infinity. Unfortunately, at present the problem of calculating the degree of the two classes $\lambda$ (the Hodge class) and $\delta$ (the boundary class) on this family presents serious difficulties. In small genus we were able to carry out some calculations via degenerations. I am planning to continue these calculations with the hope of obtaining a good lower bound on the slope. There are other candidate moving curves that may improve known slope bounds. In the near future I am planning to investigate this problem in depth.

The counterexample of Farkas and Popa has highlighted many open problems in Brill-Noether theory, the theory of representations of curves in projective space. Brill-Noether theory has been a major source of examples of divisors with small slope. Rebecca Lehman, a graduate student I am co-advising with Jason Starr, has made significant contributions to the theory by extending fundamental theorems of Eisenbud and Harris to the case of one moving ramification point. Her work raises the possibility of defining new effective divisors on the moduli space. Despite the progress, many questions of Brill-Noether theory remain open. For instance:

(1) What types and combinations of cusps can a general curve of degree $d$ and genus $g$ in $\mathbb{P}^r$ have?
(2) What kinds of secant planes can a general curve of degree $d$ and genus $g$ in $\mathbb{P}^r$ have?

Particular cases of these questions are good problems for graduate students.

### 1.3. **The cohomology of homogeneous varieties.**

The study of the cohomology of homogeneous varieties is central to mathematics, with applications to algebraic geometry, representation theory, combinatorics and the theory of symmetric functions. The ordinary Grassmannians $G(a, n)$, the $r$-step partial flag varieties $F(a_1, \ldots, a_r; n)$ and the orthogonal and symplectic Grassmannians stand out in importance among homogeneous varieties. Since Schubert cycles give an additive basis of their cohomology, the product of any two Schubert cycles $\sigma_\lambda$ and $\sigma_\mu$ may be expressed as a linear combination

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda,\mu}^\nu \, \sigma_\nu.$$

A positive combinatorial or geometric rule for determining the structure coefficients $c_{\lambda,\mu}^\nu$ (Littlewood-Richardson coefficients) is known as a Littlewood-Richardson rule. For geometric and combinatorial applications the positivity is crucial. The rich structure of these coefficients, not apparent in most presentations of the cohomology ring, are revealed by positive rules. Among the many applications of Littlewood-Richardson rules, for instance, are R. Vakil's solution of the reality of Schubert calculus [V4] and Klyachko, Knutson and Tao's solution of the Horn's conjecture [KT1].

In the case of ordinary Grassmannians there are many Littlewood-Richardson rules. The most famous are the ones in terms of Young tableaux ([Ful]), puzzles ([KT2]), and checkers ([V3]). For two-step flag varieties A. Knutson has a conjectural rule (see [BKT]) which was extended by A. Buch to three-step flag varieties. However, proving a Littlewood-Richardson rule, even in the case of two-step flag varieties, had been an outstanding open problem.

In [C5], I proved a new Littlewood-Richardson rule for ordinary Grassmannians. This rule is more general than the previously known rules: It expresses the homology class of not only intersections of Schubert varieties, but a more general class of subvarieties of Grassmannians in terms of the classes of Schubert varieties. This extension allowed me to deduce a Littlewood-Richardson rule for two-step flag varieties. Using a result of Buch, Kresch and Tamvakis ([BKT]), I also obtained the first known Littlewood-Richardson rule for the small quantum cohomology of Grassmannians. Furthermore, the method can also be used to express the class of many subvarieties of arbitrary partial flag varieties in terms of the classes of Schubert varieties.

The new Littlewood-Richardson rule is a generalization of a classical example. In the Grassmannian $G(2,4)$, we have the expression $\sigma_1^2 = \sigma_{1,1} + \sigma_2$. In the language of projective geometry, the class of the cycle of lines in $\mathbb{P}^3$ intersecting two general lines $l_1$ and $l_2$ is a sum of the classes of the cycle of lines containing a point and the cycle of lines contained in a plane. If the two lines $l_1$ and $l_2$ are general, then this relation is not apparent. However, if we specialize $l_1$ and $l_2$ to intersect at a point $p$, then any line that intersects $l_1$ and $l_2$ either contains $p$ or lies in the plane spanned by them. In this special position, the equality among the cycles becomes apparent (see Figure 1).

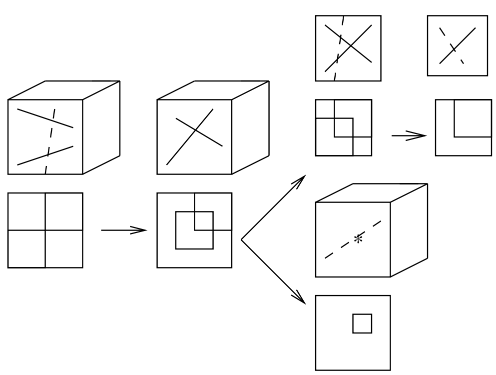

FIGURE 1. The product $\sigma_1^2$ in $G(2,4)$: Mondrian tableaux and the geometry corresponding to them.

In general, the rule is obtained by specializing the two flags defining the Schubert varieties along one-parameter families so that they become less transverse. In the process, the intersection of the two Schubert varieties breaks into two components. In fact, the rule describes not only the cohomology classes of the intersection of two Schubert varieties, but also the limiting varieties that occur as we specialize the flags. We continue the specialization with each of the components until the intersection decomposes into a union of Schubert varieties. Combinatorial objects called Mondrian tableaux provide a convenient way of encoding the ranks of intersections of the subspaces defining two flags. The process can be encoded in a game of Mondrian tableaux. The main result is as follows.

**Theorem 1.14** ([C5] Thm. 3.1)**.** *The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ of $G(a,n)$ is equal to the number of times the Mondrian tableau associated to $\sigma_\nu$ occurs in the game starting with the Mondrian tableaux $\sigma_\lambda$ and $\sigma_\mu$ in an $n \times n$ square.*

A similar rule computes the structure coefficients of the two-step flag varieties (see [C5] Thm. 4.1). An important corollary is:

**Corollary 1.15** ([C5] Thm. 5.1)**.** *The Mondrian tableaux rules for two-step flag varieties provide a quantum Littlewood-Richardson rule for Grassmannians.*

In a project, currently in progress, I am trying to address the following problem.

**Problem 1.16.** *Provide a geometric Littlewood-Richardson rule for isotropic Grassmannians.*

If one can solve this problem for orthogonal Grassmannians, then one also obtains a Littlewood-Richardson rule for the symplectic Grassmannians using the relation discussed in [BS] p.18. It is possible to formulate Littlewood-Richardson rules for orthogonal Grassmannians by reinterpreting them as Fano varieties of quadric hypersurfaces. This reinterpretation allows us to apply the same techniques that yielded the Mondrian tableaux rules to orthogonal Grassmannians [C3].

This area also provides many potential graduate and undergraduate thesis topics. For instance, the extension of these rules to other cohomology theories, such as K-theory, equivariant cohomology or equivariant K-theory, is an interesting problem (see [CV]). The combinatorial implications of these rules certainly need to be further explored. For instance, it would be interesting to see what the new rules imply about the vanishing or non-vanishing of structure coefficients.

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