There is no need to solve all the problems (a few are trickier or more time consuming than others). Five would be plenty.

1. Apply Romberg integration to compute
   \[ \int_0^1 e^x \, dx \]
   to thirty decimal places, and verify that your answer agrees with the exact answer \( e - 1 \).

2. When using a symmetric difference quotient
   \[ \frac{f(x + \epsilon) - f(x - \epsilon)}{2\epsilon} \]
   to approximate \( f'(x) \), which value of \( \epsilon \) minimizes the error if we are working to relative precision \( \delta \)? Verify that as \( \delta \to 0 \), about a third of the precision is lost. (You may assume \( f'(x) \neq 0 \) and \( f''(x) \neq 0 \).)

3. Suppose \( a_1, a_2, \ldots \) is a sequence satisfying
   \[ a_n \simeq a_\infty + \frac{C_1}{n^{\alpha_1}} + \frac{C_2}{n^{\alpha_2}} + \cdots \]
   as \( n \to \infty \), with \( 0 < \alpha_1 < \alpha_2 < \cdots \) (not necessarily integers). If \( \alpha_1, \alpha_2, \ldots \) are known but \( C_1, C_2, \ldots \) are not, then how should Richardson extrapolation be modified?

4. Find an asymptotic series for the partial sums
   \[ \sum_{i=1}^{n} \frac{1}{i} \]
   of the harmonic series, in terms of \( \log n \) and powers of \( 1/n \). The constant term is called Euler’s constant \( \gamma \), and no closed form for it is known.

5. Use Euler-Maclaurin summation to show that there exists an asymptotic series for
   \[ \sum_{i=1}^{2n} \frac{(-1)^{i-1}}{2i-1} \]
   in terms of powers of \( 1/n \), with rational coefficients (aside from the constant term), for example by writing it as
   \[ \sum_{i=1}^{n} \frac{1}{4i - 3} - \sum_{i=1}^{n} \frac{1}{4i - 1}. \]
   Prove that beyond the constant term every coefficient has a power of 2 as its denominator.
6. Make difference tables for the following sequences. What patterns do you see?

(1) 1, 1, 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, 794, 1093, 1471, \ldots
(2) 12, 11, 17, 31, 53, 153, 251, 419, 721, 1285, 2367, 4479, \ldots
(3) −1, 0, 6, 30, 114, 390, 1266, 3990, 12354, 37830, 115026, 348150, \ldots
(4) 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, \ldots
(5) 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, \ldots

Ideally you’ll identify enough information to continue the sequence indefinitely. Of course this is not a mathematically well posed problem; the goal is just to have fun looking for patterns.

7. The *Euler transform* consists of replacing a slowly converging series

\[ \sum_{k \geq 0} (-1)^k f(k) \]

with the (hopefully) much more rapidly converging series

\[ \sum_{k \geq 0} (-1)^k \frac{\Delta^k f(0)}{2^k+1}. \]

Give a very simple formal argument for the identity

\[ \sum_{k \geq 0} (-1)^k f(k) = \sum_{k \geq 0} (-1)^k \frac{\Delta^k f(0)}{2^k+1}. \]

Under which hypotheses can you prove it rigorously?

8. Prove the Taylor series expansions

\[ x \cot x = \sum_{k \geq 0} (-4)^k B_{2k} \frac{x^{2k}}{(2k)!} \]

and

\[ \tan x = \sum_{k \geq 0} (-1)^{k-1} 4^k (4^k - 1) B_{2k} \frac{x^{2k-1}}{(2k)!}, \]