18.995 Problem Set #3
(due Tuesday, October 25)

Five problems would be sufficient.

1. Let $G$ be a finite abelian group, and let $\hat{G}$ be the character group of $G$ (i.e., the group of homomorphisms from $G$ to $\mathbb{C}^\times$). Define the Fourier transform of $f: G \to \mathbb{C}$ to be the function $\hat{f}: \hat{G} \to \mathbb{C}$ given by

$$\hat{f}(\chi) = \frac{1}{|G|} \sum_{g \in G} f(g) \chi(g)$$

for $\chi \in \hat{G}$. Prove Fourier inversion:

$$f(g) = \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g).$$

2. Let $G$ be a finite abelian group, and let $H$ be a subgroup. The inclusion $H \subseteq G$ yields a restriction map from $\hat{G}$ to $\hat{H}$, and let $H^#$ be the kernel of this restriction map. (I.e., $H^#$ consists of the characters $\chi$ of $G$ for which $\chi(h) = 1$ for all $h \in H$.) Prove that for every function $f: G \to \mathbb{C}$,

$$\frac{1}{|H|} \sum_{h \in H} f(h) = \sum_{\chi \in H^#} \hat{f}(\chi).$$

3. Let $C$ be a binary linear code, i.e., a vector subspace of $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$. Let $C^\perp$ be the dual code, i.e., the orthogonal complement of $C$ under the usual inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. The weight enumerator of $C$ is defined by

$$w_C(x, y) = \sum_{i=0}^n \left| \{ z \in C : |z| = i \} \right| x^{n-i} y^i,$$

where $|z|$ denotes the Hamming weight of $z$ (i.e., the number of coordinates in which $z$ is not zero). Prove that

$$w_{C^\perp}(x, y) = \frac{1}{|C|} w_C(x + y, x - y).$$

4. Let $f$ be analytic on a disk of radius greater than $r$ about 0, and let $z_1, \ldots, z_n$ be the roots of $f$ of absolute value at most $r$ (with multiplicity). If $f(0) \neq 0$, prove that

$$\log |f(0)| = \sum_{k=1}^n \log |z_k|/r + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta.$$

5. Let $n_f(x)$ be the number of roots $f$ has of absolute value at most $x$, counted with multiplicities. In the notation from the previous problem, prove that

$$\sum_{k=1}^n \log |z_k|/r = - \int_0^r \frac{n_f(x)}{x} \, dx.$$

6. Suppose $f$ is band-limited and not identically zero, with its Fourier transform having support in $[-T, T]$, and suppose $n_f(x) \sim ax$ as $x \to \infty$ for some constant $a$. Prove that $a \leq 4T$. (I.e., $f$ can’t have too many roots without being forced to vanish identically.)

7. Show that the bound $4T$ is optimal in Problem 6, even for a Schwartz function $f$. 
