You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

**Problem 1.** Suppose that a smooth function \( u : \mathbb{R}^3 \to \mathbb{R} \) satisfies \( \Delta u \geq 6 \). Show that given \( \vec{x} \in \mathbb{R}^n \) and \( R > 0 \) the following holds.

\[
u(\vec{x}) \leq \frac{3}{5} R^2 + \frac{3}{4\pi R^3} \int_{B_R(\vec{x})} u(\vec{y}) d\vec{y}.
\]

Hint: \( \Delta |\vec{x}|^2 = 6 \).

**Problem 2.** Let \( \Omega = B_1(0) \subset \mathbb{R}^2 \). Suppose that a smooth function \( u : \bar{\Omega} \to \mathbb{R} \) satisfies

\[
\int_{\Omega} 2 \left| \nabla u \cdot \frac{(1,1)}{\sqrt{2}} \right|^2 + \left| \nabla v \cdot \frac{(1,1)}{\sqrt{2}} \right|^2 \, d\vec{x} \leq \int_{\bar{\Omega}} 2 \left| \nabla u \cdot \frac{(1,1)}{\sqrt{2}} \right|^2 + \left| \nabla v \cdot \frac{(1,-1)}{\sqrt{2}} \right|^2 \, d\vec{x},
\]

for any smooth function \( v : \bar{\Omega} \to \mathbb{R} \) such that \( u = v \) holds on \( \partial \Omega \). Show that \( 0 = 3u_{11} + 2u_{12} + 3u_{22} \) holds in \( \Omega \).

Hint: Consider \( \hat{u}(x_1, x_2) = u\left(\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}\right) \).

**Problem 3.** Let a smooth function \( u : \mathbb{R}^2 \setminus B_1(0) \to \mathbb{R} \) be harmonic. Find all solutions \( u(r \cos \theta, r \sin \theta) \) satisfying

\[
u(\cos \theta, \sin \theta) = \cos(2\theta), \quad \lim_{r \to +\infty} \frac{1}{r} u(r \cos \theta, r \sin \theta) = 0.
\]

**Problem 4.** Show that the Green function to a smooth bounded domain \( \Omega \) is symmetric. Namely,

\[
G(x, y) = G(y, x).
\]

Hint: Given \( x \neq y \), choose \( \epsilon < |x - y| \). Next, define \( u(z) = G(x, z) \) and \( v(z) = G(y, z) \), and apply the Green’s identity over the domain \( \Omega \setminus (B_\epsilon(x) \cup B_\epsilon(y)) \). By passing \( \epsilon \to 0 \), show \( u(y) = v(x) \).