Problem 1. Let \( u(x,t) \) be the smooth solution to the following Cauchy-Neumann problem:

\[
\begin{align*}
    u_t(x,t) &= u_{xx}(x,t), \quad \text{for } 0 \leq x \leq L, 0 \leq t, \\
    u_x(0,t) &= u_x(L,t) = 0, \quad \text{for } 0 \leq t, \\
    u(x,0) &= g(x), \quad \text{for } 0 \leq x \leq L,
\end{align*}
\]

where \( g(x) \) is smooth. Show the following inequality

\[
\frac{d}{dt} \int_0^L |u_x(x,t)|^2 + |u_t(x,t)|^2 dx \leq 0.
\]

Problem 2. Let \( u(x,t) \) be the smooth solution to the following Cauchy-Neumann problem:

\[
\begin{align*}
    u_t(x,t) &= u_{xx}(x,t), \quad \text{for } 0 \leq x \leq 1, 0 \leq t, \\
    u_x(0,t) &= u_x(1,t) = 0, \quad \text{for } 0 \leq t, \\
    u(x,0) &= g(x), \quad \text{for } 0 \leq x \leq 1,
\end{align*}
\]

where \( g(x) \) is smooth. Show that \( u(x,t) \) uniformly converges to the constant \( \int_0^1 g(s) ds \) as \( t \to +\infty \) by using the following steps.

1. Show that

\[
\int_0^1 |u_x(x,t)|^2 dx \leq e^{-\frac{t}{2}} \int_0^1 |g'(x)|^2 dx.
\]

2. Show that

\[
\left| u(x,t) - \int_0^1 g(x) dx \right|^2 \leq 4e^{-\frac{t}{2}} \int_0^1 |g'(x)|^2 dx.
\]

Hint: Use Lemma 3 and Theorem 4 in lecture notes.
Problem 3. Show that the following Cauchy-Dirichlet problem has a unique smooth solution, and express the solution in exact form.

\[ u_t(x, t) = u_{xx}(x, t), \quad \text{for } 0 \leq x \leq \pi, 0 \leq t, \]
\[ u(0, t) = 0, u(\pi, t) = 2\pi, \quad \text{for } 0 \leq t, \]
\[ u(x, 0) = 2x + \sin x + \sin(2x), \quad \text{for } 0 \leq x \leq \pi. \]

Hint: Consider \( v(x, t) = u(x, t) - 2x \).

Problem 4. Let \( u(x, t) \) be the smooth solution to the following Cauchy-Neumann problem;

\[ u_t(x, t) = u_{xx}(x, t), \quad \text{for } 0 \leq x \leq L, 0 \leq t, \]
\[ u_x(0, t) = u_x(L, t) = 0, \quad \text{for } 0 \leq t, \]
\[ u(x, 0) = g(x), \quad \text{for } 0 \leq x \leq L, \]

where \( g(x) \) is smooth.

1. Show that
\[ |u(x, t)| \leq \max_{0 \leq x \leq L} |g(x, 0)|. \]

2. Show that the smooth function
\[ w = \frac{1}{t + 1} |u_x|^2 + \frac{1}{2} u^2 \]
satisfies
\[ w_t \leq w_{xx} \]
for all \( (x, t) \in [0, L] \times [0, +\infty) \).

3. Establish an upper bound for \( |u_x(x, t)|^2 \) where \( t > 0 \) in terms of \( g \) and \( t \).