18.152 Midterm assignment
due April 8th 9:30 am

1. PRELIMINARY

Given a set $A \subset \mathbb{R}^n$ and a function $u : A \to \mathbb{R}$, we say that

(a) $u \in C(A)$ if $u$ is continuous in $A$,
(b) $u \in D^k(A)$ if $u$ is $k$-times differentiable in $A$,
(c) $u \in C^k(A)$ if $u$ is $k$-times differentiable and its $k$-th order derivatives are continuous in $A$,
(d) $u \in C^\infty(A)$ if $u$ is smooth ($\infty$-many times differentiable) in $A$,
(e) $u \in C^{0,1}(A)$ if $u$ is locally Lipschitz continuous in $A$, (Definition 1)
(f) $u \in C^{k,1}(A)$ if $u$ is $k$-times differentiable and its $k$-th order derivatives are locally Lipschitz continuous in $A$.

**Definition 1.** We say that $u : A \to \mathbb{R}$ is Lipschitz continuous in $A$ if there exists some constant $C_A$ such that

$$|u(x) - u(y)| \leq C_A|x - y|,$$

holds for all $x, y \in A$.

We say that $u : A \to \mathbb{R}$ is locally Lipschitz continuous in $A$ if given any compact subset $K \subset A$, $u$ is Lipschitz continuous in $K$.

We recall a version of the integration by parts.

**Theorem 2** (Integration by parts). A bounded open set $\Omega \subset \mathbb{R}^n$ has the smooth boundary $\partial \Omega$. Then,

$$\int_\Omega u_i(x)dx = \int_{\partial \Omega} u(\sigma)\nu_i(\sigma)d\sigma,$$

where $\nu_i = \langle \nu, e_i \rangle$.

**Proof.** We define $V : \Omega \to \mathbb{R}^n$ by $V(x) = u(x)e_i$. Then, the divergence theorem implies

$$\int_\Omega u_i(x)dx = \int_\Omega \text{div}V(x)dx = \int_{\partial \Omega} (V(\sigma), \nu(\sigma))d\sigma = \int_{\partial \Omega} u(\sigma)\nu_i(\sigma)d\sigma.$$

2. LAPLACE EQUATION

Let $\Omega = B_1(0) \subset \mathbb{R}^2$. Given $f \in C^{0,1}(\Omega)$, we define

\[
(4) \quad u(x) = -\int_{\Omega} G(x, y) f(y) dy.
\]

**Problem 1** (4 points). Show that

\[
(5) \quad \int_{\Omega} G(x, y) dy = \frac{1}{4} (1 - |x|^2),
\]

holds for $|x| \leq 1$.

**Problem 2** (4 points). Show that the following holds in $\Omega$,

\[
(6) \quad |u(x)| \leq \frac{1}{4} (1 - |x|^2) \sup_{\Omega} |f|.
\]

In particular, $u = 0$ on $\partial \Omega$.

**Theorem 3.** $u$ is differentiable in $\Omega$. Moreover, for each $i = 1, 2$, $\frac{\partial}{\partial x_i} u(x) = v_i(x)$ holds in $\Omega$, where $v_i(x)$ is given by

\[
(7) \quad v_i(x) = -\int_{\Omega} \frac{\partial}{\partial x_i} G(x, y) f(y) dy.
\]

**Proof.** We choose some function $\rho \in C^1(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $0 \leq \rho' \leq 2$, $\rho(t) = 0$ for $t \leq 1$ and $\rho(t) = 1$ for $t \geq 2$. Then, given $\epsilon > 0$ we define

\[
(8) \quad w_\epsilon(x) = -\int_{\Omega} [\Phi(x - y) \rho_\epsilon - \varphi(x, y)] f(y) dy.
\]

where $\rho_\epsilon = \rho(|x - y|/\epsilon)$. If $B_{2\epsilon}(x) \subset \Omega$ then $w_\epsilon \in C^4(\Omega)$ and

\[
(9) \quad |u - w_\epsilon| \leq \int_{B_{2\epsilon}(x)} \Phi(x - y) |1 - \rho_\epsilon| |f(y)| dy \leq C\epsilon^2 (1 + |\log \epsilon|) \sup_{\Omega} |f|,
\]

and

\[
(10) \quad |v_i - \frac{\partial}{\partial x_i} w_\epsilon| \leq \int_{B_{2\epsilon}(x)} \left| \frac{\partial}{\partial x_i} \Phi(x - y) (1 - \rho_\epsilon) \right| |f(y)| dy
\]

\[
\leq \sup_{\Omega} |f| \int_{B_{2\epsilon}(x)} \left| \frac{\partial}{\partial x_i} \Phi(x - y) \right| + \frac{2}{\epsilon} |\Phi(x - y)| dy
\]

\[
\leq C\epsilon (1 + |\log \epsilon|) \sup_{\Omega} |f|.
\]

Hence, in any compact subset in $\Omega$, $w_\epsilon$ and $\frac{\partial}{\partial x_i}$ uniformly converge to $u$ and $v_i$. Thus $u$ is differentiable and $\frac{\partial}{\partial x_i} u = v_i$. \qed
Problem 3 (2 point). Verify $u \in C(\Omega)$ by using Problem 2 and Theorem 3.

Given $i, j \in \{1, 2\}$, we define $v_{ij}(x)$ by

$$v_{ij}(x) = -\int_{\Omega \setminus \{x\}} \left( \frac{\partial^2}{\partial x_i \partial x_j} \Phi(x - y) \right) [f(y) - f(x)] dy$$  \hspace{1cm} (13)

$$+ \int_{\Omega} \left( \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x, y) \right) f(y) dy$$  \hspace{1cm} (14)

$$- f(x) \int_{\partial \Omega} \left( \frac{\partial}{\partial \nu_i} \Phi(x - \sigma) \right) \nu_j(\sigma) d\sigma,$$  \hspace{1cm} (15)

where $\nu_j(\sigma) = \langle \nu(\sigma), e_j \rangle = \langle \sigma, e_j \rangle = \sigma_j$.

Problem 4 (6 points). Given $i, j \in \{1, 2\}$, $x \in \Omega$ and $\epsilon > 0$ such that $B_{2\epsilon}(x) \subset \Omega$, we define

$$w_\epsilon(x) = -\int_{\Omega} \left[ \rho_\epsilon(x, y) \frac{\partial}{\partial x_i} \Phi(x - y) - \frac{\partial}{\partial x_i} \varphi(x, y) \right] f(y)$$  \hspace{1cm} (16)

where $\rho_\epsilon$ is given in the proof of Theorem 3. Then, show that there exists some constant $C$ such that

$$|u_i(x) - w_\epsilon| \leq C\epsilon, \hspace{1cm} |v_{ij}(x) - \frac{\partial}{\partial x_j} w_\epsilon| \leq C\epsilon.$$  \hspace{1cm} (17)

Hint 1: Since $f \in C^{0,1}(\overline{\Omega})$, then there exists some $C_0$ such that

$$|f(x_1) - f(x_2)| \leq C_0|x_1 - x_2|,$$  \hspace{1cm} (18)

holds for $x_1, x_2 \in \overline{\Omega}$.

Hint 2: You may use Theorem 2.

The result of Problem 4 implies

Theorem 4. $u \in D^2(\Omega)$ and $\frac{\partial^2}{\partial x_i \partial x_j} u = v_{ij}$ holds in $\Omega$.

Problem 5 (2 point). Show that $\Delta u = f$ holds in $\Omega$.

Problem 6 (2 point). Show that given $g \in C^{2,1}(\overline{\Omega})$ and $f \in C^{0,1}(\overline{\Omega})$, there exists a unique $u \in D^2(\Omega) \cap C(\overline{\Omega})$ satisfying $\Delta u = f$ in $\Omega$ and $u = g$ on $\partial \Omega$. 
Remark 5. If \( f \in C^{0,\alpha}(\overline{\Omega}) \) for \( \alpha \in (0,1) \), then we have \( u \in C^{2,\alpha}(\overline{\Omega}) \). Hence, if \( f \in C^{2,1} \) then by \( C^{2,1} \subset C^{2,\alpha} \) we have \( u \in C^{2,\alpha}(\overline{\Omega}) \) for all \( \alpha \in (0,1) \). The proof is given in [Gilbarg-Trudinger] section 4.

3. Liouville theory

**Problem 7** (6 point). Suppose that a positive function \( u \in C^{\infty}(\mathbb{R}^2 \setminus \{0\}) \) is harmonic. Show that \( u \) is a constant function.

**Problem 8** (2 point). Find a non-constant positive harmonic function \( u \in C^{\infty}(\mathbb{R}^n \setminus \{0\}) \).

**Problem 9** (6 point). Suppose that a harmonic function \( u \in C^{\infty}(\mathbb{R}^2_+) \) satisfies \( |u(x)| \leq x_2 \), where \( \mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\} \). Show that \( u(x) = cx_2 \) for some constant \( c \in [-1,1] \).

**Problem 10** (6 point). Suppose that a smooth solution \( u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) to the diffusion equation \( u_t = \Delta u + u^2 \) satisfies \( u(x,t) = u(x+e_i,t) \) for each \( i \in \{1, \cdots, n\} \). Show that \( u = 0 \).

4. Maximum principle

**Problem 11** (5 point). Let \( \Omega = B_1(0) \subset \mathbb{R}^2 \). Given a positive function \( f \in C^\infty(\overline{\Omega}) \), we suppose that a strictly convex smooth function \( u \in C^\infty(\overline{\Omega}) \) satisfies \( u = 0 \) on \( \partial B_1(0) \) and

\[
(19) \quad \det \nabla^2 u(x) = f(x),
\]

holds in \( \overline{\Omega} \), where \( \det \nabla^2 u = u_{11}u_{22} - u_{12}^2 \). Show that

\[
(20) \quad u(x) \geq -\frac{1}{2} (1 - |x|^2) \sup_{y \in \Omega} \sqrt{f(y)},
\]

holds for all \( x \in \Omega \).
Problem 12 (Bonus problem, 5 point). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Suppose that a smooth solution $u : \overline{Q_T} \to \mathbb{R}$ (where $Q_T = \Omega \times (0, T]$) to the heat equation $u_t = \Delta u$ satisfies the boundary condition $u = g$ on $\partial_p Q_T$ for some $g \in C^\infty(\Omega)$. Show that if $g$ satisfies $g \geq 0$ in $\Omega$ and $g > 0$ in $\partial \Omega$, then $u(x, t) > 0$ holds for $t > 0$. 