Proof of Problem 1. \( u(x) = \frac{1}{4}(1 - |x|^2) \in C^\infty(\Omega) \) satisfies \( \Delta u = -1 \) in \( \Omega \) and \( u = 0 \) on \( \partial\Omega \). Hence, the Green’s representation formula implies the desired result. \( \square \)

Proof of Problem 4. By Theorem 3, we have
\begin{align*}
(1) \quad u_i(x) - w_i(x) &= \int_{\Omega} (\rho_{\epsilon} - 1) \Phi_{x_i}(x - y) f(y) dy \\
(2) \quad &= \int_{B_{2\epsilon}(x)} (\rho_{\epsilon} - 1) \Phi_{x_i}(x - y) f(y) dy.
\end{align*}

Since \( |\Phi_{x_i}(x - y)| \leq C_1 |x - y|^{-1} \), we have
\begin{align*}
(3) \quad |u_i - w_i| &\leq C_2 \int_{B_{2\epsilon}(x)} |x - y|^{-1} dy = C_2 \int_{0}^{2\epsilon} \int_{0}^{2\pi} dr d\theta = C_3 \epsilon.
\end{align*}

Next, we have
\begin{align*}
(4) \quad \frac{\partial}{\partial x_j} w_i(x) &= -\int_{\Omega} \frac{\partial}{\partial x_j} \left[ \rho_{\epsilon}(x, y) \frac{\partial}{\partial x_i} \Phi(x - y) \right] (f(y) - f(x)) dy \\
&\quad + \int_{\Omega} \left( \frac{\partial^2}{\partial x_j \partial x_i} \rho_{\epsilon}(x, y) \right) f(y) dy \\
&\quad - f(x) \int_{\Omega} \frac{\partial}{\partial x_j} \left[ \rho_{\epsilon}(x, y) \frac{\partial}{\partial x_i} \Phi(x - y) \right] dy.
\end{align*}

To reformulate the last integral, we observe
\begin{align*}
(7) \quad \frac{\partial}{\partial x_j} \left( \rho_{\epsilon}(x, y) \frac{\partial}{\partial x_j} \Phi(x - y) \right) &= \frac{\partial}{\partial y} \left( \rho_{\epsilon}(x, y) \frac{\partial}{\partial y} \Phi(x - y) \right).
\end{align*}

Since \( \rho_{\epsilon}(x, y) \frac{\partial}{\partial y} \Phi(x - y) \in C^\infty(\Omega) \), Theorem 2 and \( \rho_{\epsilon}(x, y) = 1 \) on \( \partial\Omega \) yield
\begin{align*}
(8) \quad \int_{\Omega} \frac{\partial}{\partial x_j} \left[ \rho_{\epsilon}(x, y) \frac{\partial}{\partial x_i} \Phi(x - y) \right] dy &= \int_{\partial\Omega} \left[ \frac{\partial}{\partial y} \Phi(x - \sigma) \right] v_j(\sigma) d\sigma.
\end{align*}

Hence,
\begin{align*}
(9) \quad v_{ij}(x) - \frac{\partial}{\partial x_j} w_i(x) &= \int_{B_{2\epsilon}(x)} \frac{\partial}{\partial x_j} \left[ (\rho_{\epsilon} - 1) \frac{\partial}{\partial x_i} \Phi(x - y) \right] (f(y) - f(x)) dy.
\end{align*}

By using \( |\nabla \rho_{\epsilon}| \leq C \epsilon^{-1}, |\nabla \Phi| \leq C|x - y|^{-1}, |\nabla^2 \Phi| \leq C|x - y|^{-2} \), and \( |f(x) - f(y)| \leq C|x - y| \), we have
\begin{align*}
(10) \quad |v_{ij} - \frac{\partial}{\partial x_j} w_i| &\leq C_4 \int_{B_{2\epsilon}(x)} \epsilon^{-1} + |x - y|^{-1} dy \leq C_5 \epsilon.
\end{align*} \( \square \)
Proof of Problem 6. Everybody proved the existence. Hence, we show the uniqueness here.

Suppose that there exists two solutions $u, v \in D^2(\Omega) \cap C^0(\bar{\Omega})$. Then, $w = u - v \in D^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $\Delta w = 0$ in $\Omega$ and $w = 0$ on $\partial \Omega$.

We define $w_\epsilon(x) = w + \epsilon(x - |x|^2)$ and claim $w \leq w_\epsilon$ holds in $\Omega$ for all $\epsilon > 0$. If not, there exists a point $x_0 \in \Omega$ such that $w(x_0) - w_\epsilon(x_0) = \max_{\Omega}(w - w_\epsilon) > 0$. Then, $w_\epsilon = w - w_\epsilon$ attains its maximum at the interior point $x_0$, and thus $\Delta w_\epsilon(x_0) \leq 0$. However, $\Delta w_\epsilon = \Delta w - \Delta w_\epsilon = 4\epsilon > 0$. Therefore, we have $w \leq w_\epsilon$ for all $\epsilon > 0$. Passing $\epsilon \to 0$ yields $w \leq 0$ in $\bar{\Omega}$. Similarly, we obtain $w \geq 0$ in $\bar{\Omega}$, and thus $w = 0$.

First proof of Problem 7. We recall $a_0(r) = (2\pi)^{-1} \int_0^{2\pi} u(r, \theta) d\theta$ which satisfies $a_0'' + r^{-1}a_0' = 0$. Hence, $a_0 = c_1 + \log r$ for some constant $c_1, c_2$. By $a_0(r) > 0$, we have $c_2 = 0$ and thus $a_0(r) = c_1 > 0$.

Since $c_1$ is the average of $u$ on each circle $\partial B_r(0)$, given $r > 0$ there exists some angle $\theta_r$ such that $u(r, \theta_r) = c_1$. Without loss of generality, we assume $\theta_r = 0$. Then, applying the Harnack inequality for $B_{r/2}(r, 0)$, we have $C_3^{-1}c_1 \leq u(r, \theta) \leq C_3c_1$ holds in $|\theta| \leq \frac{\pi}{10}$ for some $C_3$. We apply the same argument on $B_{r/2}(r \cos \frac{\pi}{10}, r \sin \frac{\pi}{10})$ so that we have $C_3^{-2}c_1 \leq u(r, \theta) \leq C_3^2c_1$ holds in $\theta \leq [-\frac{\pi}{20}, \frac{\pi}{20}]$. We iterate this process finite times to obtain $C_4^{-1}c_1 \leq u \leq C_4c_1$ on $\partial B_r(0)$ for some $C_4$ which is independent of $r$. Namely, we have $u \leq C_5$ in $\mathbb{R}^2 \setminus \{0\}$.

Now, we recall $a_n(r) = (\pi)^{-1} \int_0^{2\pi} u(r, \theta) \cos(n\theta) d\theta$ which satisfies $a_n'' + r^{-1}a_n' - n^2r^{-2}a_n = 0$. Solving ODEs yields $a_n(r) = c_{1,n}r^{-n} + c_{2,n}r^n$. However, $|a_n| \leq C_5$ holds for all $r > 0$ and thus $a_n = 0$. Similarly, $b_n(r) = (\pi)^{-1} \int_0^{2\pi} u(r, \theta) \sin(n\theta) d\theta = 0$. Therefore, $u(r, \theta) = a_0(r) = c_1$.  

Second proof of Problem 7. We define $v : \mathbb{R}^2 \to \mathbb{R}$ by

\begin{equation}
 v(y_1, y_2) = u(e^{y_1} \cos y_2, e^{y_1} \sin y_2).
\end{equation}

Then, we can directly compute $\Delta v = e^{2y_1}\Delta u = 0$. Hence, $v$ is an entire positive harmonic function. Therefore, by the Liouville theory $v$ is a constant. Namely, $u$ is a constant.
Proof of Problem 9. Since \( u(x_1, 0) = 0 \), we have

\[
\tag{12} u(r \cos \theta, r \sin \theta) = \sum_{m=1}^{\infty} a_m(r) \sin(m \theta),
\]

where

\[
\tag{13} a_m(r) = \frac{2}{\pi} \int_0^\pi u(r \cos \theta, r \sin \theta) \sin(m \theta) d\theta.
\]

Since \( u \) is harmonic, we have

\[
\tag{14} a_m'' + \frac{1}{r} a_m' - \frac{m^2}{r^2} a_m = 0,
\]

and thus

\[
\tag{15} a_m(r) = b_m r^{-m} + c_m r^m,
\]

for some constant \( b_m, c_m \).

However, we have \( |u(x)| \leq x_2 \leq r \) and thus

\[
\tag{16} |a_m(r)| \leq \frac{2}{\pi} \int_0^\pi r d\theta = 2r.
\]

Hence, \( a_1(r) = c_1 r \) and \( a_m(r) = 0 \) for \( m \geq 2 \). Namely,

\[
\tag{17} u = \frac{2}{\pi} c_1 r \sin \theta = \frac{2c_1}{\pi} x_2.
\]

Since \( |u| \leq x_2 \), there exists some \( c \in [-1, 1] \) such that \( u = cx_2 \). \( \square \)

Proof of Problem 10. As like the problem set 2, by the divergence theorem and the Hölder inequality, \( E(t) = \int_{\Omega} u(x, t) dx \) (where \( \Omega = (-1, 1)^n \)) satisfies

\[
\tag{18} E' = \int_{\Omega} u_t dx = \int_{\Omega} \Delta u + u^2 dx = \int_{\Omega} u^2 dx \geq \left( \int_{\Omega} dx \right)^{-1} E^2 = E^2.
\]

Suppose that \( E(T) < 0 \) holds at some \( T \in \mathbb{R} \). Then, \( E(t) \leq E(T) < 0 \) for all \( t \leq T \). Hence, we can divide (18) by \( E^2 \) to obtain \( -(E^{-1})' \geq 1 \) for \( t \leq T \).

This implies

\[
\tag{19} -E^{-1}(T) = -E^{-1}(t) - \int_t^T (E^{-1})'(s) ds \geq \int_t^T ds = T - t.
\]

Passing \( t \to -\infty \) yields a contradiction, namely \( E(t) \geq 0 \) for all \( t \in \mathbb{R} \).

Next, we suppose \( E(T) > 0 \) holds at some \( T \in \mathbb{R} \). Then, \( E(t) \geq E(T) > 0 \) for all \( t \geq T \). Hence, we can divide (18) by \( E^2 \) to obtain \( -(E^{-1})' \geq 1 \) for \( t \geq T \).

This implies

\[
\tag{20} E^{-1}(T) = E^{-1}(t) - \int_t^T (E^{-1})'(s) ds \geq \int_t^T ds = t - T.
\]

Passing \( t \to \infty \) yields a contradiction, namely \( E(t) \leq 0 \) for all \( t \in \mathbb{R} \).
In conclusion, \( E(t) = 0 \) holds for all \( t \in \mathbb{R} \). Hence, (18) implies

\[
0 = E' = \int_{\Omega} u^2 \, dx,
\]

and therefore \( u = 0 \) in \( \Omega \). Since \( u \) is periodic, \( u = 0 \) in \( \mathbb{R}^n \times \mathbb{R} \). \qed

**First proof of Problem 11.** Let \( K = \sup \sqrt{f} \) and given \( \epsilon > 0 \) define

\[
w_\epsilon(x) = u(x) + \frac{1}{2}(K + \epsilon)(1 + \epsilon - |x|^2).
\]

We claim that \( w_\epsilon \geq 0 \) holds in \( B_1(0) \). If not, there exists \( x_0 \in B_1(0) \) such that \( \inf w_\epsilon = w_\epsilon(x_0) \), because \( w_\epsilon = \frac{1}{2}(K + \epsilon)\epsilon > 0 \) on \( \partial B_1(0) \).

Since \( \nabla^2 u(x_0) \) is a symmetric matrix, there exist two unit orthogonal eigenvectors \( v_1, v_2 \) and corresponding eigenvalues \( \lambda_1, \lambda_2 \). In particular, the strict convexity implies \( \lambda_1, \lambda_2 > 0 \). Since \( w_\epsilon \) attains its minimum at the interior point \( x_0 \), we have

\[
0 \leq v_i^T \nabla^2 w_\epsilon(x_0)v_i = v_i^T \left( \nabla^2 u(x_0) - (K + \epsilon)I \right) v_i = \lambda_i - (K + \epsilon),
\]

for each \( i = 1, 2 \), where \( I \) is the identity matrix. This yields a contradiction as follows.

\[
f(x_0) = \det(\nabla^2 u(x_0)) = \lambda_1 \lambda_2 \geq (K + \epsilon)^2 > K^2 = \sup f.
\]

Namely, \( w_\epsilon \geq 0 \) holds, and thus passing \( \epsilon \to 0 \) complete the proof. \qed

**Second proof of Problem 11.** We recall \( w_\epsilon \) and its interior maximum point \( x_0 \) in the first proof. Since \( \nabla^2 w_\epsilon(x_0) \) is semi-positive definite, we have

\[
0 \geq \det(\nabla^2 w_\epsilon(x_0)) = \det(\nabla^2 u(x_0) - (K + \epsilon)I)
\]

\[
= \det(\nabla^2 u(x_0)) - (K + \epsilon)\Delta u(x_0) + (K + \epsilon)^2.
\]

On the other hand, we have \( \frac{1}{2} \Delta u \geq (\det \nabla^2 u)^{\frac{1}{2}} = \sqrt{f} \). Hence,

\[
0 \geq f(x_0) - 2\sqrt{f(x_0)}(K + \epsilon) + (K + \epsilon)^2 = (K + \epsilon - \sqrt{f(x_0)})^2 \geq \epsilon^2.
\]

Namely, we have \( w_\epsilon \geq 0 \) by the contradiction above. \qed
Proof of Problem 12. Without loss of generality, we assume $\Omega \subset B_R(0) \setminus B_1(0)$. We recall the heat kernel

$$K(x, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \quad (28)$$

which is continuous in $\mathbb{R}^n \times [0, T] \setminus (0, 0)$, $K(x, 0) = 0$ for $x \neq 0$, and $K > 0$ for $t > 0$. By the continuity, there exists some constant $M$ such that

$$M = \sup_{\Omega \times [0, T]} K(x, t). \quad (29)$$

On the other hand, $g > 0$ on $\partial \Omega$ implies that there exists some $\epsilon > 0$ such that $\inf_{\partial \Omega} g = \epsilon$. Hence, $v(x, t) = \epsilon M^{-1} K(x, t)$ satisfies $v \leq u$ on $\partial \Omega Q_T$. Therefore, by the (weak) maximum principle we have $u \geq v$ in $Q_T$. In particular, we have $v = \epsilon M^{-1} v > 0$ for $t > 0$, and therefore $u > 0$ for $t > 0$. \qed