1. Maximum Principle with Neumann Boundary Condition

**Theorem 1.** \( u : [-L, L] \times [0, T) \to \mathbb{R} \) is a smooth function satisfying

\[
\begin{align*}
    u_t &\leq u_{xx}, \quad \text{for } |x| \leq L, 0 \leq t, \quad (1) \\
    u_x(-L, t) &\geq 0 \geq u_x(L, t), \quad \text{for } 0 \leq t, \quad (2) \\
    u(x, 0) &\leq 0, \quad \text{for } |x| \leq L. \quad (3)
\end{align*}
\]

Then, we have

\[
    u(x, t) \leq 0 \quad (4)
\]

for all \( |x| \leq L, 0 \leq t \).

**Proof.** Given \( \epsilon > 0 \), we define

\[
    u^\epsilon(x, t) = \epsilon \left[ x^2 + 3t + 1 \right] \quad (5)
\]

Then, we have

\[
\begin{align*}
    u^\epsilon_t &= 3 \epsilon > 2 \epsilon = u_{xx}, \quad \text{for } |x| \leq L, 0 \leq t, \quad (6) \\
    u^\epsilon_x(-L, t) &= -2 \epsilon < 0, \quad u^\epsilon_x(L, t) = 2 \epsilon > 0, \quad \text{for } 0 \leq t, \quad (7) \\
    u^\epsilon(x, 0) &\geq \epsilon > 0, \quad \text{for } |x| \leq L. \quad (8)
\end{align*}
\]

We claim \( u(x, t) < u^\epsilon(x, t) \) holds for all \( (x, t) \). Suppose that it fails. Then, since \( u^\epsilon(x, 0) > u(x, 0) \), there exists some \( (x_0, t_0) \in [-L, L] \times (0, T) \) such that \( u(x, t) < u^\epsilon(x, t) \) holds for all \( |x| \in L \) and \( t \in (0, t_0) \), and we have \( u(x_0, t_0) = u^\epsilon(x_0, t_0) \).

**Case 1.** Suppose \( |x_0| < L \).

We define \( w(x, t) = u^\epsilon(x, t) - u(x, t) \). Then, \( w(x, t) > 0 \) for \( t < t_0 \) implies \( w(x, t_0) \geq 0 \). Moreover, we have \( w(x_0, t_0) \) by definition of \( (x_0, t_0) \). Namely, \( w(x, t_0) \) attains its minimum at the interior point \( x_0 \). Hence, we have \( w_{xx}(x_0, t_0) \geq 0 \), namely

\[
    u^\epsilon_{xx}(x_0, t_0) \geq u_{xx}(x_0, t_0). \quad (9)
\]

However, \( w(x_0, t) > 0 = w(x_0, t_0) \) for \( t < t_0 \) implies

\[
    u^\epsilon_t(x_0, t_0) - u_t(x_0, t_0) = w_t(x_0, t_0) = \lim_{t \to t_0} \frac{w(x_0, t_0) - w(x_0, t)}{t_0 - t} \leq 0. \quad (10)
\]
Therefore, we have a contradiction

\[ u_t^e(x_0, t_0) > u_{xx}^e(x_0, t_0) \geq u_{xx}(x_0, t_0) \geq u_t(x_0, t_0) \geq u_t^e(x_0, t_0). \]  

(11)

**Case 2.** Suppose \(|x_0| = L\).

Without loss of generality, we consider the case \(x_0 = L\). We recall \(w(x, t_0) \geq w(x_0, t_0) = w(L, t_0)\) holds for \(|x| \leq L\). Therefore,

\[ w_x(L, t_0) = \lim_{x \to L} \frac{w(L, t_0) - w(x, t_0)}{L - x} \geq 0. \]  

(12)

This yields a contradiction to the given boundary condition.

\[ -2L = u_t^e(L, t_0) \geq u_x(L, t_0) = 0. \]  

(13)

**Corollary 2.** \(u : [-L, L] \times [0, T] \to \mathbb{R}\) is a smooth function satisfying

\[ u_t \leq u_{xx}, \quad \text{for } |x| \leq L, 0 \leq t, \]  

(14)

\[ u_x(-L, t) \geq 0 \geq u_x(L, t), \quad \text{for } 0 \leq t, \]  

(15)

\[ u(x, 0) = g(x), \quad \text{for } |x| \leq L. \]  

(16)

Then, we have

\[ u(x, t) \leq \max_{|x| \leq L} g(x). \]  

(17)

for all \(|x| \leq L, 0 \leq t\).

**Proof.** We define \(v(x, t) = u(x, t) - \max g\), and apply the maximum principle.

\[ \square \]

2. **Decay estimate**

We begin with establishing a Poincare type inequality.
Lemma 3 (1D Poincare inequality). Suppose that a smooth function $u : [0, 1] \to \mathbb{R}$ has a point $x_0 \in [0, 1]$ satisfying $u(x_0) = 0$. Then, the following holds for all $x \in [0, 1]$.

$$|u(x)|^2 \leq 4 \int_0^1 |u'(s)|^2 ds$$  \hspace{1cm} (18)

Proof. Let $u$ attain its maximum at $x_1$. Without loss of generality, we assume $x_1 \geq x_0$.

$$|u(x_1)|^2 = |u(x_1)|^2 - |u(x_0)|^2 = \int_{x_0}^{x_1} \frac{d}{ds}|u(s)|^2 ds = \int_{x_0}^{x_1} 2uu' ds.$$  \hspace{1cm} (19)

By the Arithmetic Mean-Geometric Mean inequality, we have

$$\frac{1}{2}u^2 + 2|u'|^2 \geq 2uu'.$$  \hspace{1cm} (20)

Hence,

$$|u(x_1)|^2 \leq \int_{x_0}^{x_1} \frac{1}{2}u^2 + 2|u'|^2 ds \leq \int_0^1 \frac{1}{2}u^2 + 2|u'|^2 ds \leq \frac{1}{2}|u(x_1)|^2 + 2 \int_0^1 |u'|^2 ds.$$  \hspace{1cm} (21)

Therefore, we obtain the desired result.

$$\frac{1}{2}|u(x_1)|^2 \leq 2 \int_0^1 |u'|^2 ds$$  \hspace{1cm} (22)

□

Theorem 4. $u : [0, 1] \times [0, T] \to \mathbb{R}$ is a smooth function satisfying

$$u_t = u_{xx}, \quad \text{for} \quad 0 \leq x \leq 1, 0 \leq t,$$  \hspace{1cm} (23)

$$u(0, t) = u(1, t) = 0, \quad \text{for} \quad 0 \leq t,$$  \hspace{1cm} (24)

$$u(x, 0) = g(x), \quad \text{for} \quad 0 \leq x \leq 1.$$  \hspace{1cm} (25)

Then, we have

$$\int_0^1 |u(x, t)|^2 dx \leq e^{-\frac{t}{2}} \int_0^1 |g(x)|^2 dx.$$  \hspace{1cm} (26)

Namely, $\lim_{t \to 0} \int u^2 dx = 0$.

Proof. We define an energy

$$E(t) = \int_0^1 |u(x, t)|^2 dx.$$  \hspace{1cm} (27)
Then,
\[
\frac{d}{dt} E(t) = \int_0^1 2uu_t dx = \int_0^1 2uu_{xx} dx = 2uu_x|_0^1 - 2 \int_0^1 u_x^2 dx = -2 \int_0^1 u_x^2 dx.
\]  
(28)
Since \( u(0, t) = u(1, t) = 0 \), we can apply the lemma above so that
\[
\frac{d}{dt} E(t) \leq -\frac{1}{2} \max_{0 \leq x \leq 1} |u(x, t)|^2 = -\frac{1}{2} \int_0^1 \max_{0 \leq x \leq 1} |u(x, t)|^2 dx \leq -\frac{1}{2} E(t).
\]  
(29)
Therefore,
\[
\frac{d}{dt} \left( e^{\frac{t}{2}} E(t) \right) = e^{\frac{t}{2}} E'(t) + \frac{1}{2} e^{\frac{t}{2}} E(t) \leq 0.
\]  
(30)
This gives the desired result.
\[
e^{\frac{t}{2}} E(t) \leq \int_0^1 g^2(x) dx.
\]  
(31)
\[ \square \]

3. Review: Fourier series

We recall the Fourier series. In this class, we will use the following fact without proofs.

Given a smooth function \( f : [-L, L] \to \mathbb{R} \) with \( f(-L) = f(L) \), the following holds
\[
\lim_{N \to +\infty} \sup_{|x| \leq L} |f(x) - S_N(x)| = 0,
\]
for the partial sums \( S_N(x) \) of Fourier series,
\[
S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x/L) + \sum_{m=1}^{\infty} b_m \sin(m\pi x/L),
\]
where
\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx, \quad a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{m\pi x}{L} \right) dx, \quad b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{m\pi x}{L} \right) dx.
\]
Suppose that \( f : [0, L] \to \mathbb{R} \) is a smooth function satisfying \( f(0) = 0 \). Then,
\[
\lim_{N \to +\infty} \sup_{0 \leq x \leq L} |f(x) - S_N(x)| = 0,
\]
holds for the partial sums \( S_N(x) \) of Fourier sine series,
\[
S_N(x) = \sum_{m=1}^{\infty} b_m \sin(m\pi x/L),
\]
(31)
where

\[
b_m = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{m\pi x}{L} \right) dx.
\]

Suppose that \( f : [0, L] \to \mathbb{R} \) is a smooth function satisfying \( f'(0) = 0 \). Then,

\[
\lim_{N \to +\infty} \sup_{0 \leq x \leq L} |f(x) - S_N(x)| = 0,
\]
holds for the partial sums \( S_N(x) \) of Fourier cosine series,

\[
S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x/L),
\]
where

\[
a_0 = \frac{2}{L} \int_0^L f(x) dx, \quad a_m = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{m\pi x}{L} \right) dx.
\]

4. Review: ODE

We recall the some well-known results in ODEs. We will also use them without proofs.

Suppose that a function \( u(x) \) satisfies the following differential equation

\[
u''(x) + \mu^2 u(x) = 0.
\] (32)

Then,

\[
u(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x),
\] (33)

for some constants \( c_1, c_2 \) depending on initial (or boundary data). For example, if \( u(x) \) satisfies \( u(0) = 0 \) and \( u'(0) = 1 \), then the constants must be \( c_1 = \mu^{-1} \) and \( c_2 = 0 \).

Suppose that a function \( u(x) \) satisfies the following differential equation

\[
u'(x) = \lambda u(x).
\] (34)

Then,

\[
u(x) = ce^{\lambda x},
\] (35)
for some constant $c$ depending on the initial data.