Chern classes of Heegner divisors

Lecture 11 • Speaker: Kartik Prasanna • Notes by: Charlotte Chan • April 1, 2016

This is a write-up of a very nice talk given by Kartik Prasanna. The purpose of this talk is to explain Chapter 5 of [Br] and explain the motivation behind [Br].

We are interested in studying cycles on Shimura varieties attached to orthogonal groups $O(V)$, where $V$ is a quadratic space over $\mathbb{Q}$ with form $Q$ of signature $(2, n)$. (More generally, we can consider, for example, a quadratic space over a totally real field $F$ that has signature $(2, n)$ at $\infty$ and is definite at all other infinite places.) Then, via the embeddings $O(2, l) \hookrightarrow O(2, l)$, the Heegner divisors will give rise to cycles of every codimension in the Shimura variety for $O(2, n)$.

Pick a lattice $L \subseteq V$. For simplicity, we assume $L$ is unimodular. Let $\Gamma(L) \subset O(V)$ be the stabilizer of $L$ in $O(V)$; this is a discrete subgroup. Recall that we have an $n$-dimensional symmetric space $\mathbb{H}$ associated to $L$ and that the quotient $X_L = \mathbb{H} / \Gamma(L)$ is a connected component of the orthogonal Shimura variety. Recall also that for each negative integer $m$ such that $m \in \mathbb{Q}(V)$, we have an associated Heegner divisor $H(m) \in CH^1(X_L)$. Once and for all, set $\kappa := 1 + n/2 > 0$ and $k := 2 - \kappa = 1 - n/2 < 2$. Then the Borcherds lift in [B2] is a given by a map

$$M_k^!(SL_2 \mathbb{Z}) \longrightarrow \{\text{mero aut form for } \Gamma(L)\}$$

$$f \longmapsto \Psi_f,$$

where $M_k^!(SL_2 \mathbb{Z})$ denotes the space of weakly holomorphic modular forms of weight $k$ (i.e. $f$ may have poles at the cusps but is otherwise holomorphic). Writing

$$f = \sum_{n > -\infty} c(n)q^n,$$

we get a meromorphic automorphic form $\Psi_f$ of weight $c(0)/2$ with divisor

$$\text{div } \Psi_f = \sum_{m < 0} c(m)H(m).$$

This is the Borcherds lift. Note that to define the Borcherds lift, we need to assume that $c(m) \in \mathbb{Z}$ for $m < 0$. We also remark that the divisors $H(m)$ were defined as divisors on $\mathbb{H}$ and were later shown to be $\Gamma(L)$-invariant. Hence the Heegner divisors $H(m)$ can be viewed as divisors on the quotient $X_L$.

**Remark** 1. What do we mean by an automorphic form? We can think of an automorphic form as a section of an automorphic vector bundle on $X_L$. The Shimura variety $X_L$ carries a family a Hodge structures of weight 2 with Hodge numbers $h^{2,0} = 1$, $h^{1,1} = n$, $h^{0,2} = 1$. There is an automorphic line bundle $\omega$ over $X_L$ that corresponds to taking the first step in the Hodge filtration, and the Borcherds lift $\Psi_f$ is a section of $\omega^{\otimes c(0)/2}$.

The following result of Borcherds gives a criterion for what can appear as the principal part of a weakly holomorphic modular form. (Note that the stated result below is a simplification of the general statement, which does not require $L$ to be unimodular.)

**Theorem 1.** There exists an $f \in M_k^!(SL_2 \mathbb{Z})$ with principal part $\sum_{-\infty < n < 0} c(n)q^n$ if and only if $\sum_{n < 0} c(n)a(-n) = 0$ for all cusp forms $\sum_{n > 0} a(n)q^n \in S_k(SL_2 \mathbb{Z})$.  

1
Example 2. Consider the special case when $L = H \oplus H$ is the sum of two copies of the hyperbolic plane. Then $V = L \otimes \mathbb{R}$ has signature $(2, 2)$ and we have $\kappa = 2$ and $k = 0$. The group $\text{SO}(V)$ is (roughly) $\text{SL}_2 \times \text{SL}_2$ and so the associated symmetric space is (roughly) a product of two copies of the upper-half plane $\mathfrak{h}$. Consider the modular $j$-function

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots$$

and consider the cusp form

$$f := j - 744 \in M^0_k(\text{SL}_2 \mathbb{Z}).$$

Then the Borcherds lift $\Psi_f$ is a form of weight 0 on $\mathfrak{h} \times \mathfrak{h}$ and

$$\Psi_f(z_1, z_2) = j(z_1) - j(z_2).$$

So, the divisor is just the diagonal $\mathfrak{h} \xrightarrow{\Delta} \mathfrak{h} \times \mathfrak{h}$. We can say something more. Since $S_2(\text{SL}_2 \mathbb{Z}) = 0$, then by Theorem 1, the principal part of a weakly holomorphic function of weight $k$ can be anything. In particular, for any linear combination of Heegner divisors on $\mathfrak{h} \times \mathfrak{h}$, there exists a meromorphic automorphic form for $\Gamma(L)$ with this specified divisor.

The main question that Bruinier answers in [Br] is:

Given a Heegner divisor, can one find an explicit section of the corresponding line bundle with this specified divisor?

To do this, we need to enlarge $M^0_k(\text{SL}_2 \mathbb{Z})$ to the space of weakly harmonic Maass forms. This space is spanned by the Poincaré series $F_m$ (discussed in Lectures 4 & 5), which are not weakly holomorphic. It will turn out that we can define lifts of $F_m$ analogous to the Borcherds lifts of weakly holomorphic modular forms, and that these lifts will have divisor exactly equal to the Heegner divisor $H(m)$. This is the construction given in [Br] (see also Lectures 6 & 7 of this seminar) and we review the construction of this lift now.

We can define a regularized theta lift (using a different regularization to the one in [B2])

$$F_m \mapsto \Phi_m,$$

where $\Phi_m$ is a $\Gamma(L)$-invariant function on $\mathbb{H}$ that is real-valued and real-analytic outside $H(m)$. As in Lecture 7, there is a decomposition

$$\Phi_m = \psi_m + \xi_m,$$

where $\xi_m$ is real-analytic on $\mathbb{H}$ and $\psi_m$ has the form

$$\log |\Psi_m(z)| = -\frac{1}{4} (\psi_m(z) - C_m)$$

for some holomorphic function $\Psi_m$ on $\mathbb{H}$ and some specified constant $C_m$. Note however that $\psi_m$ and $\xi_m$ are not $\Gamma(L)$-invariant. We have

$$\text{div}(\Psi_m) = \pi^* H(m),$$

where $\pi : \mathbb{H} \to \mathbb{H}/\Gamma(L)$.

Lemma 3. Suppose $f$ is a meromorphic function on $\mathbb{H}$ whose divisor is $\pi^* H(m)$. Let

$$J(\gamma, z) = \frac{f(\gamma z)}{f(z)}.$$

Then to give a hermitian metric on the line bundle $\mathcal{L}(H(m))$ associated to the Heegner divisor $H(m)$ is the same as giving a $C^\infty$ function $h : \mathbb{H} \to \mathbb{R}^+$ satisfying

$$h(\gamma z) = |J(\gamma, z)| \cdot h(z).$$
Furthermore, a representative for \(c_1(\mathcal{L}(H(m)))\) is given by \(\partial\bar{\partial}\log|h(z)|\).

In our situation, \(f = \Psi_m\) so that \(J(\gamma, z) = \Psi_m(\gamma z)/\Psi_m(z)\). Hence we want to find \(h\) such that \(h(z) \cdot |\Psi_m(z)|\) is \(\Gamma(L)\)-invariant. But we already know that
\[
|\Psi_m(z)| \cdot e^{-\frac{1}{4} \xi_m(z)}
\]
is \(\Gamma(L)\)-invariant, and so we can take
\[
h(z) = e^{-\frac{1}{4} \xi_m(z)}.
\]
Now the Chern class \(c_1(\mathcal{L}(H(m)))\) is given by
\[
\partial\bar{\partial} \log e^{-\frac{1}{4} \xi_m(z)} = -\frac{1}{4} \partial\bar{\partial} \xi_m(z) = -\frac{1}{4} \partial\bar{\partial} \Phi_m,
\]
and \(\Phi_m\) is \(\Gamma(L)\)-invariant. So to summarize what we’ve done, we’ve explicitly constructed a section \(\Psi_m\) of the line bundle, and moreover, we know the Chern class of this line bundle is given by \(-\frac{1}{4} \partial\bar{\partial} \Phi_m\). So \(\Phi_m\) is a Green’s function for the divisor \(H(m)\).

The point of all this is that in arithmetic intersection theory, one wants to write down not only the divisor but also a Green’s function. Bruinier’s Borcherds lift gives rise to such a function (and we moreover know a lot of explicit information about this function: it’s Fourier expansion, etc.).

Remark 2. For higher codimension, one expects that the lift should come from \(\text{Sp}_{2n}(\mathbb{Z})\) for \(n > 1\). There is very little known about how to construct the Green’s functions.

There is another thread of this story that relates Borcherds lifts to Kudla–Millson theory. We may pick up this thread in the summer.

References