Due date: 5/14/2015
Collaboration on homework is not allowed

Problem 1. Let $M$ be a random $n \times n$ matrix whose entries are $\{-1, 1\}$ independently with probability $1/2$. Use Sperner’s lemma to prove that the probability of $M$ being singular is $O(\frac{1}{\sqrt{n}})$.

Problem 2. Let $F \subseteq 2^{[n]}$ be an intersecting family. Prove that there exists an intersecting family $F'$ of size $2^n - 1$ that contains $F$.

Problem 3. Let $F \subseteq 2^{[n]}$ be a family of sets such that $|F \cap F'|$ is even for all $F, F' \in F$. Prove that $|F| \leq 2^{n/2}$.

Problem 4. Let $F \subseteq 2^{[n]}$ be a family of sets for which there are no distinct $F, F' \in F$ satisfying $F \subseteq F'$. Let $a_k$ be the number of sets of size $k$ in $F$. Prove that

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1.$$

Problem 5. Let $F \subseteq \binom{[n]}{r}$. Prove that

$$\frac{|\partial F|}{\binom{n}{r-1}} \geq \frac{|F|}{\binom{n}{r}}.$$