LECTURE 8. EXTREMAL SET THEORY II

1. Erdős-Ko-Rado theorem

Let $X$ be a finite set. A family $\mathcal{F} \subseteq 2^X$ is intersecting if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. The following theorem is an easy exercise and was first observed by Erdős, Ko, and Rado.

**Theorem 1.** If $\mathcal{F} \subseteq 2^X$ is an intersecting family, then $|\mathcal{F}| \leq 2^{|X|} - 1$.

**Proof.** Define $n = |X|$. Consider the $2^{n-1}$ pairs $\{A, \overline{A}\}$ for $A \in 2^X$. By definition, $\mathcal{F}$ may contain only at most one set from each such pair. Therefore $|\mathcal{F}| \leq 2^{n-1}$. □

There are many families achieving this bound. For example, one can take all the sets containing a fixed element. In fact, there are many different examples. Erdős, Ko, and Rado proved a similar result for families of sets of same size.

**Theorem 2.** (Erdős-Ko-Rado 1961) Suppose that $X$ is a finite set of size $n$ satisfying $n \geq 2k$. If $\mathcal{F} \subseteq \binom{X}{k}$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

The original proof used the shifting method.

**Definition 3.** For a finite set $X$, a family $\mathcal{F} \subseteq 2^X$ and two distinct elements $x, y \in X$, define $S_{x,y}(\mathcal{F}) = \{S_{x,y}(F) : F \in \mathcal{F}\}$ where

$$S_{x,y}(F) = \begin{cases} F' & y \in F, x \notin F, F' = (F \setminus \{y\}) \cup \{x\} \notin \mathcal{F} \\ F & \text{otherwise} \end{cases}$$

It is known that shifting has the following properties.

**Proposition 4.** Shifting defined above has the following properties.

(i) $|S_{x,y}(F)| = |F|$.

(ii) $|S_{x,y}(\mathcal{F})| = |\mathcal{F}|$.

(iii) If $\mathcal{F}$ is intersecting, then so is $S(\mathcal{F})$.

**Proof.** Properties (i) and (ii) immediately follow from the definition.

To prove Property (iii), let $F$ and $F'$ be distinct sets in $\mathcal{F}$. If $x \in F$ and $y \notin F$ we say that $S_{x,y}$ shifts $F$. If $F$ and $F'$ are both shifted by $S_{x,y}$ or both not shifted by $S_{x,y}$, then we clearly have $F \cap F' \neq \emptyset$. Suppose that $F$ is shifted but $F'$ is not. Since $(F \cap F') \setminus (S_{x,y}(F) \cap S_{x,y}(F')) \subseteq \{y\}$, the only way we may have $S_{x,y}(F) \cap S_{x,y}(F') = \emptyset$ is if $F \cap F' = \{y\}$. If $F \cap F' = \{y\}$, then since $F$ is shifted, we have $x \notin F$ and $x \in S_{x,y}(F)$. On the other hand, since $y \in F'$ and $F'$ is not shifted, we must have $x \in F'$. However, since $F'$ is not shifted, we have either (a) $x \in S_{x,y}(F')$ or (b) $F'' = (F' \setminus \{y\}) \cup \{x\} \in \mathcal{F}$. □
Case (a) implies that \( x \in F \cap S_{x,y}(F') \) and case (b) cannot happen since \( F'' \cap F = \emptyset \) contradicts the fact that \( F \) is intersecting.

We now present the first proof of the Erdős-Ko-Rado theorem.

**Proof of EKR 1.** If \( n = 2k \), then we can pair the set and its complement to prove that \( |F| \leq \frac{1}{2}(\binom{2k}{k}) = \binom{2k-1}{k-1} \). Suppose that \( n \geq 2k + 1 \). Without loss of generality, we may assume \( X = [n] \).

For \( i = 1, 2, \ldots, n \), define \( F_0 = F \) and \( F_i = S_{i,n}(F_{i-1}) \). Note that \( |F_{n-1}| = |F| \) and \( F_{n-1} \) is intersecting. Define \( G = \{ F \in F_{n-1} : n \notin F \} \) and \( H = \{ F \setminus \{n\} : n \in F \in F_{n-1} \} \). It suffices to show that \( H \) is intersecting, since then by induction, we have

\[
|F| \leq |G| + |H| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}.
\]

Suppose that \( H \) is not intersecting. Then there exists \( H_1, H_2 \in H \) such that \( H_1 \cap H_2 = \emptyset \). Since \( H_1 \cup H_2 \leq 2(k-1) < n-1 \), there exists an element \( i \notin H_1 \cup H_2 \) such that \( i \leq n-1 \). Consider \( F = H_1 \cup \{i\} \in F_{n-1} \). Since \( F \) is not in the image of \( S_{j,n} \) for all \( j \), this implies that \( F \in F_j \) for all \( j \). However, this means that \( (F \setminus \{n\}) \cup \{i\} = H_1 \cup \{i\} \in F_{i-1} \). On the other hand, similarly as above we have \( H_2 \cup \{n\} \in F_j \) for all \( j \). This contradicts the fact that \( F_{i-1} \) is an intersecting family.

**Proof of EKR 2.** Consider a permutation \( \sigma \in S_n \) chosen uniformly at random. Place the elements of \( X \) in a circle according to the order given by \( \sigma \).

For each \( Z \in F \), let \( A_Z \) be the event that the elements of \( Z \) form an interval on this circle and \( 1_Z \) be the indicator random variable of the event \( A_Z \).

Since \( F \) is an intersecting family and \( n \geq 2k \), for each permutation \( \sigma \), there can only be at most \( k \) sets in \( F \) whose elements form an interval on the circle. Therefore

\[
\mathbb{E} \left[ \sum_{Z \in F} 1_Z \right] \leq k.
\]

On the other hand for a fixed \( Z \in F \), the probability that the elements in \( Z \) form an interval is exactly \( k!(n-k)! / (n-1)! = \binom{n}{k} \). Therefore

\[
\frac{n}{(k)} |F| \leq k,
\]

and it follows that \( |F| \leq \frac{1}{\binom{k-1}{n-k}} \).

It is essential to have \( n \geq 2k \), since if \( n \leq 2k - 1 \), then all two sets in \( \binom{n}{k} \) are intersecting. The theorem is tight since we can take all sets containing a fixed element. Unlike in the previous case where we had several different families achieving the extremal bound, it is known that this ‘star construction’ is the unique example if \( n \geq 2k + 1 \).
A family $F$ is $t$-intersecting if for each distinct $F, F' \in F$ such that $|F \cap F'| \geq t$. Improving on previous results, Wilson proved the following theorem using a linear algebraic technique.

**Theorem 5.** (Wilson 1984) Suppose that $n > (t + 1)(k - t + 1)$. If $F$ is a $t$-intersecting family, then $|F| \leq \binom{n-1}{k-1}$.

The bound $n > (t + 1)(k - t + 1)$ is best possible. If $n \leq (t + 1)(k - t + 1)$, then we can consider the family $F = \{ |X \cap [t + 2r]| \geq t + r : X \in \binom{[n]}{k} \}$ for some $r > 0$ to obtain a $t$-intersecting family with more than $\binom{n-1}{k-1}$ sets. The combinatorial proofs do not supply a stability type result for Erdős-Ko-Rado theorem. Friedgut used discrete Fourier analytical technique to prove the following stability result. A star $S \subseteq \binom{[n]}{k}$ is a family of the form $\{ X : i \in X \in \binom{[n]}{k} \}$ for some $i \in [n]$.

**Theorem 6.** (Friedgut 2008) For each fixed positive real $\xi$, there exists $c$ such that the following holds for sufficiently large $n$. If $k \in (\xi n, (\frac{1}{2} - \xi)n)$ and $F \subseteq \binom{[n]}{k}$ is an intersecting family satisfying $|F| \geq (1 - \varepsilon)\binom{n-1}{k-1}$, then there exists a star $S$ such that $|F \setminus S| < c\varepsilon \binom{n}{k}$.

There are Erdős-Ko-Rado type results for other discrete structures as well. For example, it has been studied for intersecting family of permutations. Recently, Ellis, Friedgut, and Pilpel proved the corresponding result for permutations. Another interesting case is when we consider a family of graphs. A family $G$ of graphs with the same vertex set is triangle-intersecting if for all $G, G' \in G$, the intersection graph $G \cap G'$ contains a triangle. The following theorem confirms a conjecture of Simonovits and Sós.

**Theorem 7.** (Ellis, Filmus, Friedgut) Let $G$ be a triangle-intersecting family of graphs on vertex set $[n]$. Then $|G| \leq \frac{1}{8} \cdot 2^{\binom{n}{2}}$.

Note that by taking all graphs considering a fixed triangle, we can easily find a triangle-intersecting family of size $\frac{1}{8} \cdot 2^{\binom{n}{2}}$. These proofs use Fourier analytic technique.

### 2. Kruskal-Katona theorem

**Definition 8.** For a family $F \subseteq 2^{[n]}$, we define the shadow $\partial F = \{ F' : F' \subseteq F \in F \}$.

For a positive real number $x$, define $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$.

**Theorem 9.** Let $F \subseteq \binom{[n]}{k}$ be a given family of size $|F| = \binom{x}{k}$. Then $|\partial F| \geq \binom{x}{k-1}$.

We will use the shifting operation. For a family $F \subseteq 2^{[n]}$, define $F_0 = \{ F \in F : 1 \notin F \}$ and $F_1 = \{ F \in F : 1 \in F \}$. 

### 2. Kruskal-Katona theorem
Proposition 10. For every $\mathcal{F} \subseteq \binom{[n]}{k}$, there exists a family $\mathcal{G} \subseteq \binom{[n]}{k}$ such that

(i) $|\mathcal{F}| = |\mathcal{G}|$,
(ii) $|\partial \mathcal{G}| \geq |\partial \mathcal{F}|$, and
(iii) $G \cup \{1\} \in \mathcal{G}$ for all $G \in \partial \mathcal{G}_0$.

Proof. We need the following claim.

Claim. For all $i \in [n]$ and $i \neq 1$, the relation $\partial S_{1,i}(\mathcal{F}) \subseteq S_{1,i}(\partial \mathcal{F})$ holds.

Consider $F \in \mathcal{F}$. If $(i \notin F) \text{ or } (1, i \in F)$, then $S_{1,i}(F) = F$ and $S_{1,i}(\partial F) = \partial F$, so it is clear that $\partial F \subseteq S_{1,i}(\partial F)$. Hence it suffices to consider the case when $i \in F$ and $1 \notin F$. Define $F' = (F \setminus \{i\}) \cup \{1\}$. If $F' \in \mathcal{F}$, then $S_{1,i}(F) = F$ and $S_{1,i}(\partial F) = \partial F$. Therefore $\partial F \subseteq S_{1,i}(\partial F)$ holds.

Finally if $F' \notin \mathcal{F}$, then $S_{1,i}(F) = F'$. First note that $F' \setminus \{n\} = F \setminus \{i\} \in \partial F$ and $S_{1,i}(F \setminus \{1\}) = F \setminus \{1\}$. Hence it suffices to consider $F' \setminus \{j\} \in \partial F$ for $j \neq 1$. However, note that either $F' \setminus \{j\} = S_{1,i}(F \setminus \{j\})$ or $F' \setminus \{j\}$ is already in the family $\partial F$.

Now repeatedly apply $S_{1,i}$ for $i = 2, \ldots, n$ to the family $\mathcal{F}$. Note that $|\mathcal{F}|$ remains the same and $|\partial \mathcal{F}|$ is non-decreasing under this operation. The process eventually ends since each non-trivial operation increases the number of sets in $\mathcal{F}$ that contain the element 1. Moreover, when the operation ends, Property (iii) is satisfied since otherwise we can further continue. □

The proof of Theorem 9 follows.

Proof of Theorem 9. Suppose that $\mathcal{F} \subseteq \binom{[n]}{k}$ is a family of size $|\mathcal{F}| = \binom{x}{k}$ satisfying (iii) of the previous proposition. Suppose that $|\mathcal{F}_1| < \binom{x-1}{k-1}$. Then

$$|\mathcal{F}_0| = |\mathcal{F}| - |\mathcal{F}_1| > \binom{x-1}{k}.$$ 

However, by property (iii) above and induction, this implies that

$$|\mathcal{F}_1| \geq |\partial \mathcal{F}_0| \geq \binom{x}{k-1},$$

which is a contradiction. Therefore $|\mathcal{F}_1| \geq \binom{x-1}{k-1}$.

Define $\mathcal{G} = \{F \setminus \{1\} : F \in \mathcal{F}_1\}$. Then

$$|\partial \mathcal{F}| \geq |\mathcal{G}| + |\partial \mathcal{G}|,$$

and therefore by $|\mathcal{G}| = |\mathcal{F}_1| \geq \binom{x-1}{k-1}$ and the inductive hypothesis,

$$|\partial \mathcal{F}| \geq \binom{x-1}{k-1} + \binom{x-1}{k-2} = \binom{x}{k-1}.$$ □

The theorem above is tight when $x$ is an integer. It is a simplified version of Kruskal and Katona’s theorem proved by Lovász. The original Kruskal-Katona theorem proved independently by Kruskal (1963) and Katona (1966) can be proved using an almost identical proof but is more technical to state.
First note that every integer $m$ can be uniquely written in the following form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$$

for positive integers satisfying $a_k \geq a_{k-1} \geq \cdots \geq a_s \geq s \geq 1$.

**Theorem 11. (Kruskal-Katona)** If $\mathcal{F} \subseteq \binom{[n]}{k}$ satisfies

$$|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s},$$

for positive integers satisfying $a_k \geq a_{k-1} \geq \cdots \geq a_s \geq s \geq 1$, then

$$|\partial \mathcal{F}| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}.$$

The theorem is tight for all values of $m = |\mathcal{F}|$. Suppose that $m = (a_k) + (a_{k-1}) + (a_s)$. Then take the family

$$\mathcal{F} = \left\{ \binom{a_k}{k} \right\} \cup \left\{ a_k + 1 \cup X : X \in \binom{a_{k-1}}{k-1} \right\} \cup \left\{ a_k + 1, a_{k-1} + 1 \cup X : X \in \binom{a_{k-2}}{k-2} \right\} \cup \left\{ a_k + 1, \cdots, a_{s+1} + 1 \cup X : X \in \binom{a_s}{s} \right\}.$$ 

It is not too difficult to check that

$$\partial \mathcal{F} = \left\{ \binom{a_k}{k-1} \right\} \cup \left\{ a_k + 1 \cup X : X \in \binom{a_{k-1}}{k-2} \right\} \cup \left\{ a_k + 1, a_{k-1} + 1 \cup X : X \in \binom{a_{k-2}}{k-3} \right\} \cup \left\{ a_k + 1, \cdots, a_{s+1} + 1 \cup X : X \in \binom{a_s}{s-1} \right\}.$$ 

Therefore we see that Kruskal and Katona’s theorem is tight.

The *colexicographical order* on $\{0, 1\}^n$ is a total order defined as $v < w$ for two vectors if $v_i < w_i$ where $i$ is the maximum coordinate in which the two vectors $v$ and $w$ differ. The extremal example in Kruskal and Katona’s theorem can be alternatively described as the $m$ smallest elements in $\binom{[n]}{k}$ according to the colexicographical order. For example if $m = \binom{n}{k}$ for some integer $k$, then the $m$ smallest elements in $\binom{[n]}{k}$ is simply $\binom{[n]}{k}$.


**Proof of Erdős-Ko-Rado theorem.** Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family of size $|\mathcal{F}| > \binom{n-1}{k-1}$. Define $\mathcal{G} = \{ n \} \setminus \{ F : F \in \mathcal{F} \} \subseteq \binom{n}{n-k}$. Then $|\mathcal{G}| = |\mathcal{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Therefore the shadow of $\mathcal{G}$ in $\binom{[n]}{k}$ has size greater than
\[ \binom{n-1}{k}. \]

Since \( \mathcal{F} \) is intersecting, it follows that no set in \( \mathcal{F} \) is in the shadow of \( G \). Therefore
\[
|\mathcal{F}| + \binom{n-1}{k} \leq \binom{n}{k}.
\]

However, this implies that \( |\mathcal{F}| \leq \binom{n-1}{k-1} \) contradicting our assumption. Therefore, \( |\mathcal{F}| \leq \binom{n-1}{k-1} \). \( \square \)

Note that the uniqueness result for Erdős-Ko-Rado follows from the uniqueness for Kruskal and Katona (although the proof above does not really give the uniqueness for Kruskal and Katona’s theorem).

### 2.2. Stability

Keevash gave an alternative proof of Lovász’s version of Kruskal and Katona’s theorem. For an \( r \)-uniform hypergraph \( G \), let \( K_{r+1}^{(r)}(G) \) be the number of copies of \( K_r^{(r)} \) in \( G \). The following theorem is equivalent to the (weak version of) Kruskal and Katona’s theorem.

**Theorem 12.** If \( G \) is an \( r \)-uniform hypergraph with at most \( \binom{x}{r} \) edges, then \( K_{r+1}^{(r)}(G) \leq \binom{x}{r+1} \).

Note that \( G \subseteq K_r^{(r)}(\partial G) \). Therefore if \( |\partial G| = \binom{y}{r-1} \), then \( e(G) \leq \binom{y}{r} \). In other words, if \( e(G) = \binom{y}{r} \), then \( |\partial G| \geq \binom{x}{r-1} \). The other direction can be seen similarly.

**Definition 13.** Let \( G \) be an \( r \)-uniform hypergraph and \( v \in V(G) \) be a vertex. The link hypergraph \( L(v) \) is the \((r-1)\)-uniform hypergraph on \( V(G) \setminus \{v\} \) where \( e' \) is an edge if and only if \( e' \cup \{v\} \) is an edge of \( G \).

Note that the link hypergraph generalizes the concept of neighborhood of a vertex for a graph (i.e. when \( r = 2 \)).

**Proof from Keevash’s paper.** We argue by induction on \( r \). The base case \( r = 1 \) is trivial. We assume that the degree \( d(v) \) is non-zero for every vertex \( v \). Note that for a set \( S \) of size \( r \), the set \( S \cup \{v\} \) spans a \( K_r^{(r)}(v) \) in \( G \) if and only if \( S \) is an edge of \( G \) and spans a \( K_r^{(r-1)}(v) \) in the link hypergraph \( L(v) \). The first condition gives the estimate \( K_r^{(r)}(v) \leq e(G) - d(v) \) and the second \( K_r^{(r)}(v) \leq K_r^{(r-1)}(L(v)) \). We claim that \( K_r^{(r)}(v) \leq (\frac{x}{r} - 1)d(v) \) for every \( v \) where equality holds only if \( d(v) = (\frac{x}{r} - 1) \). To see this, suppose first that \( d(v) \geq (\frac{x}{r} - 1) \). Then by the first inequality, we see that \( K_r^{(r)}(v) \leq (\frac{x}{r} - d(v)) \leq (\frac{x}{r} - 1)d(v) \). On the other hand, if \( d(v) \leq (\frac{x}{r} - 1) \), then suppose that \( d(v) = (\frac{x}{r} - 1) \) for some \( r \leq x \). By the induction hypothesis, we know that \( K_r^{(r-1)}(L(v)) \leq (\frac{x}{r-1}) = (\frac{x}{r} - 1)d(v) \). Therefore
the claim holds. Now
\[(r + 1)K_{r+1}^{(r)}(G) = \sum_v K_{r+1}^{(r)}(v) \leq \left(\frac{x}{r} - 1\right) \sum_v d(v)\]
\[= \left(\frac{x}{r} - 1\right) r|V(G)| = (x - r)\left(\frac{x}{r}\right) = (r + 1)\left(\frac{x}{r + 1}\right).\]

Therefore \(K_{r+1}^{(r)}(G) \leq \left(\frac{x}{r+1}\right)^r\), as required. Equality holds only when all vertices have degree \((x-1)/(r-1)\). If \(G\) has \(n\) vertices, then \(n(x-1)/(r-1) = \sum_v d(v) = r^r(x/r) = x^{r-1}/(r-1)\), so \(n = x\) and \(G = K_x^{(r)}\). □

Keevash’s proof also gives a stability result for Kruskal and Katona’s theorem and thus for Erdős-Ko-Rado theorem.

3. **Isoperimetric inequalities**

Quote from wikipedia:

In mathematics, the isoperimetric inequality is a geometric inequality involving the square of the circumference of a closed curve in the plane and the area of a plane region it encloses, as well as its various generalizations. Isoperimetric literally means "having the same perimeter".

For a graph \(G\) and a subset of vertices \(X\), define the vertex boundary of \(X\) in \(G\) as \(\partial X = \{y \in V(G) \setminus X : y \text{ is adjacent to some } x \in X\}\). Define the edge boundary of \(X\) in \(G\) as the set \(\{(x,y) \in E(G) : x \in X, y \in V(G) \setminus X\}\). Isoperimetric inequality for graphs is a popular topic of study. There are two types of isoperimetric problems for graphs, vertex isoperimetric problems and edge isoperimetric problems.

A hypercube \(Q_n\) is a graph on vertex set \(\{0,1\}^n\) where two vertices are adjacent if and only if they differ in exactly one coordinate. We study isoperimetric inequality for hypercubes. More precisely, we are interested in the following questions:

What is the minimum vertex boundary over all sets of fixed size? (vertex isoperimetric problem) What is the minimum edge boundary over all sets of fixed size? (edge isoperimetric problem)

Kruskal and Katona’s theorem is closely related to the vertex isoperimetric problem. In fact, it can be considered as the isoperimetric problem for the subgraph of the hypercube induced on two consecutive levels.

3.1. **Edge isoperimetric inequality.** [From Petr Gregor’s lecture notes]

For a set \(S\), define \(\phi_n(S)\) as the size of the edge boundary of \(S\) in \(Q_n\). Define \(\phi_n(m) = \min_{S:|S|=m} \phi_n(S)\) as the minimum size of the edge boundary of a subset of \(m\) vertices in \(Q_n\).

For an integer \(m\), define \(f_n(m) = \max\{e(Q_n[S]) : S \subseteq V(Q_n), |S| = m\}\) be the maximum number of edges spanned by \(m\) vertices. Since \(Q_n\) is an
Let $\phi_n(m)$ be equal to the edge boundary of the set $X_{n,m}$ of $m$ smallest numbers in the colexicographical order of $\{0,1\}^n$. Note that $\phi_n(m)$ is equal to the number of ones in the binary expansion of $m$. We note that the extremal example for Kruskal-Katona theorem is also given by the same example (restricted to a single level).

We claim that the following lemma holds. [Proof is left as an exercise]

**Lemma 14.** For all $1 \leq k \leq \ell$, $\sum_{i=0}^{k-1} h(i) + k \leq \sum_{i=k}^{\ell-1} h(i)$.

We now prove that $f_n(m) \leq \phi_n(X_{n,m})$ by induction on $m$. The conclusion is trivially true for $m = 1$. Let $S$ be a set of size $|S| = m$ for some $m \geq 2$. Without loss of generality, we may assume that $S$ contains a vector whose $n$-th coordinate has value 0, and a vector whose $n$-th coordinate has value 1. Let $S_0 = \{v \in Q_{n-1} : (v,0) \in S\}$ and $S_1 = \{v \in Q_{n-1} : (v,1) \in S\}$. Furthermore, we may assume that $|S_1| \leq |S_0|$ and therefore $0 < |S_1| \leq |S_0| < n$. Note that

$$e(Q_n[S]) \leq e(Q_{n-1}[S_0]) + e(Q_{n-1}[S_1]) + |S_1|$$

$$\leq f_{n-1}(m_0) + f_{n-1}(m_1) + |S_1|$$

$$\leq \sum_{i=0}^{m_0-1} h(i) + \sum_{i=0}^{m_1-1} h(i) + m_1$$

$$\leq \sum_{i=0}^{m-1} h(i) + \sum_{i=0}^{m_1-1} h(i) + m_1 - \sum_{i=0}^{m_0+m_1-1} h(i) \leq m \sum_{i=1}^{m-1} h(i).$$

Therefore

$$\phi_n(m) = nm - 2 \sum_{i=0}^{m-1} h(i).$$

It implies the following more simple form

$$\phi_n(m) \geq m(n - \log_2 m).$$

The equality holds if $m = 2^k$ for some $k$.

### 3.2. Vertex isoperimetric inequality.

The vertex isoperimetric inequality for hypercubes is known as Harper’s theorem.
For two vectors $v, w \in \{0, 1\}^n$, define the Hamming distance $d(v, w)$ between $v$ and $w$ as the number of coordinates in which $v$ and $w$ differ. For a vector $v \in \{0, 1\}^n$, a Hamming sphere with center $v$ is a set $B$ satisfying

$$\{w \in \{0, 1\}^n : d(v, w) \leq r\} \subseteq B \subseteq \{w \in \{0, 1\}^n : d(v, w) \leq r + 1\}.$$ 

The minimum vertex boundary in a hypercube is achieved by a Hamming sphere.

**Theorem 15.** (Füredi-Frankl) Let $A$ and $B$ be subsets of $\{0, 1\}^n$ where

$$d(A, B) = \min\{d(A, B) : A \in A, B \in B\} = d.$$ 

Then there exist Hamming spheres $A_0$ with center 0 and $B_0$ with center 1 such that $|A_0| = |A|$ and $|B_0| = |B|$ and $d(A_0, B_0) \geq d(A, B)$.

Given a set $X \subseteq \{0, 1\}^n$ that minimizes the vertex boundary, we can apply the theorem above to the pair of sets $(X, \{0, 1\}^n \setminus X)$ to show that the minimum vertex boundary is achieved by a Hamming sphere.

**Proof sketch.** We identify vectors in $\{0, 1\}^n$ with subsets of $[n]$. Consider the sets of pairs

$$\{(A, A^*) : A \in A, A^* \notin A, |A| < |A^*|\}$$

and

$$\{(B, B^*) : B \in B, B^* \notin B, |B| > |B^*|\}.$$ 

If there are no such pairs, then $A$ is centered at 1, and $B$ is centered at 0. Otherwise, choose a pair $(A, A^*)$ or $(B, B^*)$ with minimal symmetric difference $|A \Delta A^*|$ and $|B \Delta B^*|$. Assume that the minimal pair is $(A_0, A_0^*)$. Define

$$U = A_0 - A_0^* \quad \text{and} \quad V = A_0^* - A_0.$$ 

Consider the two operations

$$U(A) = \begin{cases} 
A - U + V & \text{if } U \subseteq A, V \cap A = \emptyset, A - U + V \notin A, \\
A & \text{otherwise}
\end{cases}$$

and

$$D(B) = \begin{cases} 
B - V + U & \text{if } V \subseteq B, U \cap B = \emptyset, B - V + U \notin A, \\
B & \text{otherwise}
\end{cases}.$$ 

One can prove that

$$d(U(A), D(B)) \geq d(A, B)$$

and thus repeated applications of these operations will lead to two Hamming spheres. \qed