LECTURE 6. EMBEDDING LARGE GRAPHS

So far we studied embedding of small fixed graphs. We now study embedding of large graphs, graphs whose number of vertices are comparable or as large as the host graph.

1. Matching

A matching of a graph is a set of vertex disjoint edges. The size of a matching is the number of edges in the matching. Let \( G \) be a bipartite graph with two parts \( A \cup B \) satisfying \( |A| = |B| \). A perfect matching of \( G \) is a matching of size \( |A| = |B| \).

**Theorem 1.** (Hall 1935) Let \( G \) be a bipartite graph with two parts \( A \cup B \) where \( |A| = |B| \). If \( |N(X)| \geq |X| \) holds for all \( X \subseteq A \), then \( G \) contains a perfect matching.

**Proof.** Let \( M \) be a matching of maximum size and suppose that \( |M| < |A| \). Color the edges in two colors red/blue where the edges of \( M \) are blue and all other edges are red. An alternating path is a path of length at least 2 in which the color of the edges alternate.

Let \( A' = A \cap V(M) \) and \( B' = B \cap V(M) \). If \( N(B') = A' \), then the induced subgraph on \( (A \setminus A') \cup (B \setminus B') \) satisfies the condition. Hence we can find a perfect matching by induction.

Otherwise let \( a \notin A' \) be a vertex adjacent to some vertex in \( B \). Let \( X \subseteq A \) and \( Y \subseteq B \) be sets of vertices that intersect some alternating path starting at \( a \). If some alternating path starting at \( a \) ends in a red edge, then we can switch the red edges and blue edges along this path to find a matching of size \( |M| + 1 \) but this contradicts our choice of \( M \).

Hence for all such alternating paths, all vertices other than \( a \) are adjacent to a blue edge. Therefore if \( M' \subseteq M \) is the set of blue edges that are contained in some alternating path starting at \( a \), then \( V(M') \cap B = Y \) and \( (V(M') \cap A) \cup \{a\} = X \). This implies that \( |X| > |Y| \). From the given condition, it follows that there exists an edge \( \{x, z\} \) satisfying \( x \in X \) and \( x \notin Y \). Since the blue edge intersecting \( x \) also intersects \( Y \), the edge \( \{x, z\} \) cannot be a blue edge. Thus we found an alternating path starting at \( a \) and ending in a red edge. As seen above, this implies that we can find a matching of size \( |M| + 1 \).

Note that the condition is necessary and sufficient since a bipartite graph with a perfect matching clearly satisfies \( |N(X)| \geq |X| \) for all \( X \subseteq A \). The proof easily implies the following result.

Corollary 2. Let $G$ be a bipartite graph with two parts $A \cup B$. If $|N(X)| \geq |X|$ for all $X \subseteq A$ of size $|X| \geq k$, then there exists a matching of size $k$ between $A$ and $B$.

A necessary and sufficient condition for general (not necessarily bipartite) graphs are also known. An odd component of a graph is a connected component of odd size.

Theorem 3. (Tutte) A graph has a perfect matching if and only if $G - U$ has at most $|U|$ odd components for each $U \subseteq V$.

We will not prove Tutte's theorem in this class. The following theorem follows as a simple corollary.

Theorem 4. If an $n$-vertex graph has minimum degree at least $\frac{n}{2}$, then it contains a perfect matching.

Note that the minimum degree $\frac{n}{2}$ is tight. It can be seen in two ways. First for odd $n$, consider a bipartite graph with $\frac{n+1}{2}$ vertices in one part and $\frac{n-1}{2}$ vertices in the other part. Second for odd $n$, consider two complete graphs on $\frac{n+1}{2}$ vertices sharing one vertex in common. We will later provide a direct proof of the theorem.

2. Long paths and cycles

The length of a path and a cycle is measured in terms of its number of edges. We investigate different types of conditions that implies the existence of long paths and cycles in a given graph.

Theorem 5. (Pósa) If $G$ has minimum degree at least $k \geq 1$, then it contains a path of length at least $k$. If $k \geq 2$, then $G$ contains a cycle of length at least $k + 1$.

Proof. Consider the longest path $P = (v_0, v_1, \cdots, v_\ell)$ of $G$. All neighbors of $v_0$ must be in $V(P)$ as otherwise we can find a path longer than $P$. This in particular implies that $|V(P)| \geq \deg(v) + 1 \geq k + 1$. Moreover, there exists an index $i \geq \deg(v) \geq k$ such that $v_i$ is a neighbor of $v_0$. If $i \geq 2$, then $(v_0, v_1, \cdots, v_i, v_0)$ is a cycle of length $i + 1 \geq k + 1$. □

The condition above is tight since $K_k$ is a graph of minimum degree $k - 1$ having no path of length at least $k$, and no cycle of length $k + 1$.

A Hamilton cycle of a graph is a cycle that contains all vertices of the graph. If the minimum degree is sufficiently large compared to the total number of vertices, then we can in fact find a Hamilton cycle.

Theorem 6. (Dirac 1952) If $G$ is an $n$-vertex graph of minimum degree at least $\frac{n}{2}$, then $G$ contains a Hamilton cycle.

Proof. The given condition implies that $G$ is connected. Consider the longest path $P = (v_0, v_1, \cdots, v_\ell)$ of $G$. As in the proof above, all neighbors of $v_0$ are in $V(P)$ and all neighbors of $v_\ell$ are in $V(P)$. Define $I = \{v_{i-1} : v_i \in N(v_0)\}$
Proof. Consider the longest path $G$ then each pivot point $v$ and note that $|I| + \frac{n}{2} \geq n > \ell$. Since $|I| + \frac{n}{2} \geq n > \ell$, there exists an index $i$ such that $v_i \in N(v_0)$ and $v_{i-1} \in N(v_i)$. Note that $(v_0, v_i, v_{i+1}, \cdots, v_{\ell}, v_{i-1}, v_{i-2}, \cdots, v_0)$ is a cycle $C$ of length $\ell$. Assume that $C$ is not a Hamilton cycle. Then since $G$ is connected, there exists a vertex $x \notin V(C)$ that is adjacent to some vertex in $C$. However this implies that we can find a path that is longer than $P$ and contradicts our choice of $P$. Therefore $C$ is a Hamilton cycle.

Note that Dirac’s theorem implies Theorem ??-?? The two graphs showing the tightness of Theorem ??-?? shows that the minimum degree $\frac{n}{2}$ is tight (non-balanced complete bipartite graphs and two complete graphs sharing a vertex).

The following conjecture was made as a generalization of Dirac’s theorem. For a graph $G$, the $k$-th power of $G$ denoted $G^{(k)}$ is the graph obtained from $G$ by connected all vertices that are within distance $k$ of each other (in the graph distance defined by $G$). (Pósa conjectured the $k = 2$ case)

**Conjecture 7.** (Pósa 1962, Seymour 1974) For all $n \geq k \geq 2$, if an $n$-vertex graph has minimum degree at least $(1 - \frac{1}{k})n$, then it contains a $k$-th power of Hamilton path as a subgraph.

The following result can be found in Lovász’s book, and is attributed to Pósa. For a vertex subset $X$, define $N(X)$ as the set of vertices incident to at least one vertex in $X$. Note that $N(X)$ and $X$ are not necessarily disjoint.

**Theorem 8.** (Pósa 1976) If $G$ is a graph where each $X \subseteq V(G)$ with $|X| \leq k$ satisfies

$$|N(X) \setminus X| \geq 2|X| - 1,$$

then $G$ contains a path of length at least $3k - 2$.

**Proof.** Consider the longest path $P = (v_0, v_1, \cdots, v_\ell)$ of $G$. Note that for each $v_i \in N(v_0)$, the path $(v_{i-1}, \cdots, v_0, v_i, \cdots, v_\ell)$ is another path of length $\ell$. We say that $v_{i-1}$ is a new endpoint obtained by rotating the path $P$ with pivot point $v_i$ and fixed endpoint $v_\ell$. Let $I = \{v_0, v_{i_1}, \cdots, v_{i_t}, v_\ell\}$ be the set of endpoints obtained by repeatedly rotating the path $P$ (we consider all rotations in both directions, i.e., those with fixed endpoint $v_0$ and those with fixed endpoint $v_\ell$). Note that $N(I) \setminus I \subseteq \{v_1, v_{i_1+1}, \cdots, v_{i_t+1}, v_{\ell-1}\}$ since otherwise we can find a new endpoint or find a path that is longer than $P$. Therefore $|N(I) \setminus I| \leq 2|I| - 2$ from which it follows that $|I| > k+1$. Consider an arbitrary set $I' \subseteq I$ of size $|I'| = k$. Then

$$|V(P)| \geq |I'| + |N(I') \setminus I'| \geq k + (2k - 1) = 3k - 1,$$

and therefore $P$ has length at least $3k - 2$. \hfill \Box

The tightness of the theorem above can be seen from a complete graph on $3k - 1$ vertices. The technique used in the proof is known as the Pósa rotation-extension technique.
3. Trees: Friedman-Pippenger theorem

Friedman and Pippenger extended Pósa’s theorem, Theorem ??, to trees.

**Theorem 9.** (Friedman-Pippenger 1987) If $G$ is a graph where each $X \subseteq V(G)$ with $|X| \leq 2k - 2$ satisfies

$$|N(X)| \geq (d + 1)|X|,$$

then $G$ contains every tree with $k$ vertices and maximum degree at most $d$.

**Proof.** Throughout the proof, we fix $k, d,$ and $k$. Call a graph $G$ expanding if $|N(X)| \geq (d + 1)|X|$ for all $|X| \leq 2k - 2$. We call a tree $T$ small if it has at most $k$ vertices and maximum degree at most $d$. We will define a class of “good embeddings” of a small tree into an expanding graph satisfying the following properties:

1. If $T$ consists of a single vertex and $G$ is an expanding graph, then there is a good embedding of $T$ in $G$.
2. If $T$ is a small tree and $S$ is a subtree of $T$ obtained by deleting a leaf vertex, then any good embedding of $S$ into an expanding graph $G$ can be extended to a good embedding of $T$ into $H$.

Note that the theorem follows once we successfully define the class of good embeddings. Let $f : V(T) \to V(G)$ be an embedding of $T$ in $G$.

- For a set $X \subseteq V(G)$, define the assets of $X$ under $f$ as $A_f(X) = |N_G(X) \setminus f(V(T))|$.
- For a vertex $x \in V(G)$, define $J_f(x)$ as the degree of $f^{-1}(x)$ in $T$ if $x \in f(T)$, and 0 otherwise.
- Define $B_f(x) = d - J_f(x)$ for each $x \in V(G)$.
- For a set $X \subseteq V(G)$, define the liability of $X$ under $f$ as $B_f(X) = \sum_{x \in X} B_f(x)$.
- For a set $X \subseteq V(G)$, define the balance of $X$ under $f$ as $C_f(x) = A_f(X) - B_f(X)$.
- A set $X \subseteq V(G)$ is solvent, critical, and bankrupt under $f$ if $C_f(X) \geq 0$, $C_f(X) = 0$, and $C_f(X) < 0$, respectively.
- An embedding $f$ of $T$ in $G$ is good if every $X \subseteq V(G)$ with $|X| \leq 2k - 2$ is solvent.

Property 1 can be easily verified. Let $T$ be a single vertex graph, and $G$ be an expanding graph. Let $X \subseteq V(G)$ be an arbitrary set of size at most $2k - 2$. Then

$$A_f(X) = |N_G(X) \setminus f(V(T))| \geq |N_G(X)| - |f(V(T))| \geq |N_G(X)| - (d + 1)|X| \geq d|X|.$$

On the other hand $J_f(x) = 0$ for all $x \in V(G)$ and therefore $B_f(x) = d|X|$. Hence $A_f(X) - B_f(X) \geq 0$, showing that $X$ is solvent.

Let $T$ be a small tree and $S$ be a tree obtained form $T$ by removing a leaf. Let $f$ be a good embedding of $S$ in $G$. The following lemmas establish some properties of critical sets.

**Lemma 10.** If $X$ is critical under $f$ and $|X| \leq 2k - 2$, then $|X| \leq k - 1$. 
Proof. Since $B_f(x) \leq d$ for all vertices $x \in V(G)$, we have $B_f(X) \leq d|X|$. On the other hand since $|V(S)| \leq k - 1$, $A_f(X) = |N_G(X) \setminus f(V(S))| \geq (d + 1)|X| - (k - 1)$. If $X$ is critical, then $C_f(X) = A_f(X) - B_f(X) = 0$ and thus $d|X| \geq (d + 1)|X| - (k - 1)$, and it follows that $|X| \leq k - 1$. \qed

Lemma 11. If $X, Y$ are critical under $f$ and $|X|, |Y| \leq k - 1$, then $X \cup Y$ is critical under $f$ and $|X \cup Y| \leq k - 1$.

Proof. The definition immediately implies that $B_f(X) + B_f(Y) = B_f(X \cup Y) + B_f(X \cap Y)$. Furthermore it is not too difficult to check that $A_f(X) + A_f(Y) \geq A_f(X \cup Y) + A_f(X \cap Y)$. Therefore it follows that $C_f(X) + C_f(Y) \geq C_f(X \cup Y) + C_f(X \cap Y)$.

Note that if $|X|, |Y| \leq k - 1$, then $|X \cup Y|, |X \cap Y| \leq 2k - 2$. Hence by definition of good embedding, we must have $C_f(X \cup Y) \geq 0$ and $C_f(X \cap Y) \geq 0$. Therefore if $C_f(X) = C_f(Y) = 0$, then $C_f(X \cup Y) = C_f(X \cap Y) = 0$, implying that $X \cup Y$ is critical under $f$. Moreover, by the previous lemma, it follows that $|X \cup Y| \leq k - 1$. \qed

Let $v$ be the deleted leaf of $T$, and $w$ be the node to which $v$ is connected. Hence $w \in V(S)$. Let $1, 2, \cdots, t$ be the neighbors of $f(w)$ that are not in $f(V(S))$. Since $\{f(w)\}$ is solvent and $B_f(f(w)) \leq d - 1$, we see that $t = A_f(f(w)) \geq 2$. For each $i = 1, 2, \cdots, t$, let $g_i$ be the extension of $f$ obtained by defining $g_i(v) = i$. If $g_i$ is a good embedding for some $i$, then Property 2 holds and thus we may assume that $g_i$ is not a good embedding for each $i$. Thus for each $i$, there exists a set $X_i$ of size $|X_i| \leq 2k - 2$ that is bankrupt under $g_i$, i.e., $C_{g_i}(X_i) < 0$. Since $A_{g_i}(X_i) \geq A_f(X_i) - 1$, $B_{g_i}(X_i) \leq B_f(X_i)$, and $C_f(X_i) \geq 0$, we must have $C_{g_i}(X_i) = 0$, $A_{g_i}(X_i) = A_f(X_i) - 1$, and $B_{g_i}(X_i) = B_f(X_i)$. The first equality implies $|X_i| \leq k - 1$ (by the lemma above), the second implies $i \in N(X_i)$, and the third implies $f(w) \notin X_i$ since $J_{g_i}(f(w)) = J_f(f(w)) + 1$.

Define $X^* = \bigcup_{i=1}^t X_i$ and $X' = X^* \cup \{f(w)\}$. Since $N(X') \setminus N(X^*) \subseteq N(f(w))$, it follows that $A_f(X') = A_f(X^*)$. Moreover since $w$ has degree at most $d - 1$ in $S$, it follows that $B_f(f(w)) \geq 1$, and thus $B_f(X') > B_f(X')$. Hence $C_f(X') = A_f(X') - B_f(X') < A_f(X) - B_f(X) = C_f(X) = 0$, and we see that $X'$ is bankrupt, contradicting the fact that $f$ is a good embedding. \qed

Bollobás made the following conjecture on spanning trees.
Conjecture 12. (Bollobás) For all positive real number \( \varepsilon < \frac{1}{2} \) and positive integer \( \Delta \), if \( n \) is sufficiently large, then every \( n \)-vertex graph of minimum degree at least \( \left( \frac{1}{2} + \varepsilon \right)n \) contains all trees on at most \( n \) vertices with maximum degree at most \( \Delta \).

4. Perfect \( H \)-packing

Another simple yet interesting structure is \( H \)-packing. For a graph \( H \), an \( H \)-packing of \( G \) is a set of vertex-disjoint copies of \( H \). A perfect \( H \)-packing that cover each vertex of \( G \) exactly once.

Theorem 13. (Corrádi-Hajnal 1963) For all \( n \) divisible by 3, if an \( n \)-vertex graph has minimum degree at least \( \frac{2}{3}n \), then it contains a perfect triangle-packing.

The generalization of the above theorem to general cliques was made by Erdős and solved by Hajnal and Szemerédi.

Theorem 14. (Hajnal-Szemerédi 1970) Let \( k \) be a positive integer. For all \( n \) divisible by \( k \), if an \( n \)-vertex graph has minimum degree at least \( (1 - \frac{1}{k})n \), then it contains a perfect \( K_k \)-packing.

The minimum degree condition is best possible. To see this, consider a graph on \( n = mk \) vertices and partition its vertex set into two parts \( X \cup Y \), where \( |X| = m + 1 \) and \( |Y| = (m - 1)k - 1 \). Add all edges incident to \( Y \) and note that the graph has minimum degree slightly less than \( (1 - \frac{1}{k})n \). Each copy of \( K_k \) contains at most one vertex from \( X \), and thus we cannot have a perfect \( K_k \)-packing. We will not cover the proof in this class.

An equitable vertex-coloring of a graph \( G \) is a coloring of its vertex set for which the number of vertices in each color class differs by at most 1. Hajnal and Szemerédi’s theorem has the following equivalent form (exercise: check that the two theorems are indeed equivalent).

Theorem 15. (Hajnal-Szemerédi) If \( G \) is an \( n \)-vertex graph of maximum degree at most \( \Delta \), then there exists an equitable proper vertex-coloring using \( \Delta + 1 \) colors.

What about for general graphs \( H \)? What is the minimum degree threshold?

5. Connection to Ramsey theory

For a graph \( H \), define its Ramsey number \( r(H) \) as the minimum integer \( n \) such that for every red/blue edge coloring of \( K_n \), there exists a monochromatic copy of \( H \). The following bounds on the Ramsey number of complete graphs are well-known:

\[
2^{k/2} \leq r(K_k) \leq 4^k.
\]

Recall that a graph \( H \) is \( d \)-degenerate if all its subgraphs contains a vertex of degree at most \( d \). In 1975, Burr and Erdős made the following conjecture and initiated the study of Ramsey number of sparse graphs.
Conjecture 16. For all positive integers \( d \), there exists \( c = c(d) \) such that every \( d \)-degenerate graph \( H \) satisfies \( r(H) \leq c|V(H)| \).

Note the striking contrast between sparse graphs and complete graphs (linear versus exponential dependency). Proving linear upper bounds on Ramsey number of a graph is closely connected to embedding large graphs since we are trying to find a subgraph whose number of vertices is comparable to the number of vertices of the host graph. Burr and Erdős’s conjecture is an important conjecture in Ramsey theory and is still open.

In 1983, Chvátal, Rödl, Szemerédi, and Trotter proved the following relaxed version of the conjecture.

Theorem 17. (Chvátal, Rödl, Szemerédi, Trotter 1983) For all positive integer \( \Delta \), there exists \( c = c(\Delta) \) such that every graph \( H \) of maximum degree at most \( \Delta \) satisfies \( r(H) \leq c|V(H)| \).

Their proof was based on the regularity lemma. The following is the main lemma of the proof.

Lemma 18. For all \( \delta, \Delta, r \), there exists \( \varepsilon \) such that the following holds. Let \( G \) be a graph with vertex partition \( V = V_1 \cup \cdots \cup V_r \) whose \((\varepsilon, \delta)\)-reduced graph is the complete graph and \( |V_i| = n \) for all \( i \in [r] \). If \( H \) is an \( r \)-partite graph of maximum degree at most \( \Delta \) such that \( |V(H)| = m < (\delta - \varepsilon)^r n \), then \( G \) contains a copy of \( H \).

**Proof.** Let \( x_1, x_2, \ldots, x_m \) be an arbitrary enumeration of the vertices of \( H \).

Since \( H \) is an \( r \)-chromatic graph, we will color \( H \) using \([r]\) and then embed the vertices of color \( i \) into \( V_i \). We will embed the vertices one vertex at a time in the order of this enumeration. Let \( f \) denote the partial embedding that we update throughout the process. Thus after the \( t \)-th step, \( f \) will be a partial embedding of the subgraph of \( H \) induced on \( \{x_1, x_2, \ldots, x_t\} \).

For each \( i > t \), define \( N_i(x_i) = N(x_i) \cap \{x_1, \ldots, x_t\} \). For each \( i > t \) we will maintain a set \( U_i^{(t)} \) each \( u \in U_i^{(t)} \) is adjacent to \( f(x) \) for all \( x \in N_i(x_i) \) such that

\[
|U_i^{(t)}| \geq (\delta - \varepsilon)|N_i(x_i)|n.
\]

Hence if \( f(x_{t+1}) \) is chosen as a vertex in \( U_i^{(t)} \), then \( f \) will be a partial embedding of \( H[\{x_1, \ldots, x_{t+1}\}] \).

We now show how to define \( f(x_{t+1}) \) given \( f \) defined on \( \{x_1, \ldots, x_t\} \) and satisfying the above. While selecting \( f(x_{t+1}) = u \in U_i^{(t)} \), we have to define sets \( U_i^{(t+1)} \) for \( i > t + 1 \) satisfying the properties above. Naturally, we will define

\[
U_i^{(t+1)} = \begin{cases} U_i^{(t)} \cap N(u) & \text{if } x_i \text{ is adjacent to } x_{t+1} \\ U_i^{(t)} & \text{otherwise} \end{cases}
\]

Note that (\ref{eq:property}) is automatically satisfied for \( i > t + 1 \) for which \( x_i \) is not adjacent to \( x_{t+1} \) since \( N_{t+1}(x_i) = N(x_i) \). For \( i > t + 1 \) for which \( x_i \) is
adjacent to \( x_{t+1} \), we must choose \( u \) so that
\[
|U_i^{(t)} \cap N(u)| \geq (\delta - \varepsilon)|U_i^{(t)}|.
\]
For notational convenience, assume that \( x_{t+1} \) must get embedded to \( V_1 \) and \( x_i \) to \( V_2 \). Since \( |U_i^{(t)} \cap N(u)| \geq (\delta - \varepsilon)n > \varepsilon n \), by the regularity of the pair \((V_1, V_2)\), there are at most \( \varepsilon n \) vertices in \( V_1 \) for which \( |U_i^{(t)} \cap N(u)| < (\delta - \varepsilon)|U_i^{(t)}| \). There are at most \( \Delta \) vertices \( x_i \) adjacent to \( x_{t+1} \), there are at most \( \varepsilon \Delta n \) vertices that we must avoid for \( u \). Since \( |U_i^{(t)}| \geq (\delta - \varepsilon)n > \varepsilon \Delta n \), we see that there exists a vertex \( u \in U_i^{(t)} \) for which \( f(x_{t+1}) = u \) will satisfy all the constraints that we imposed.

Next is a simple proposition stating the regularity is preserved under taking complements.

**Proposition 19.** Let \( G \) be a graph and \((V_1, V_2)\) be an \( \varepsilon \)-regular pair in \( G \). Then \((V_1, V_2)\) is an \( \varepsilon \)-regular pair in the complement of \( G \).

The proof of Chvátal, Rödl, Szemerédi, Trotter theorem follows.

**Proof.** Let \( c \) be a large constant depending on \( \varepsilon = 4^\Delta \). Let \( H \) be a \( n \)-vertex graph of maximum degree at most \( \Delta \). Consider a red/blue edge coloring of \( K_{cn} \). Let \( G \) be the red graph and consider an \( \varepsilon \)-regular partition \( V_0 \cup V_1 \cup \cdots \cup V_k \) of \( G \) where \( k > 4^\Delta \). Note that it also is an \( \varepsilon \)-regular partition of the blue graph by the Proposition above.

Let \( \Gamma \) be the \((\varepsilon, 0)\)-reduced graph of the partition. By the exercise in the problem set, we may assume that \( \Gamma \) has minimum degree at least \((1 - \varepsilon)k \). Color the edge \( \{i, j\} \in E(R) \) with red if \((V_i, V_j)\) is \( \varepsilon \)-regular with red density at least \( 1/2 \), and blue if \((V_i, V_j)\) is \( \varepsilon \)-regular with blue density at least \( 1/2 \). By Turán’s theorem, we can find a clique of size at least \( K_{4^\Delta - 1} \) in \( R \). Hence by Ramsey’s theorem, \( \Gamma \) contains a monochromatic copy of \( K_\Delta \). Then by the lemma above, if \( c \) is large enough, then we can find a copy of \( H \) in the red graph or the blue graph.

\[\square\]

### 6. Almost \( H \)-packing

Suppose that we have a graph with vertex partition \( V_1 \cup V_2 \cup V_3 \) where \((V_i, V_j)\) are \( \varepsilon \)-regular with sufficiently large density for each distinct pair \((i, j)\) and \(|V_1| = |V_2| = |V_3| = m \). We can repeatedly remove triangles across the partition until less than \( \varepsilon' m \) vertices remain in each part. This is true since if there are more vertices, then the subgraph induced on the remaining vertices is \( \frac{\varepsilon'}{\varepsilon} \)-regular. Alon and Yuster and used this idea to prove the following theorem.

**Theorem 20.** (Alon-Yuster 1992) For every \( \varepsilon > 0 \) and \( h \), there exists \( n_0 = n_0(\varepsilon, h) \) such that for every graph \( H \) with \( h \) vertices and for every \( n > n_0 \), any graph \( G \) with \( n \) vertices and minimum degree at least \( \frac{\chi(H) - 1}{\chi(H)} \) contains at least \( (1 - \varepsilon)\frac{n}{h} \) vertex-disjoint copies of \( H \).
Alon and Yuster then conjectured a strengthened version of this theorem.

**Conjecture 21.** For every $h$, there exists a constant $c(h)$ such that for every graph $H$ with $h$ vertices, every graph $G$ with $n$ vertices and minimum degree at least $\chi(H) - 1$ contains at least $n - c(h)$ vertex-disjoint copies of $H$.

Note that the conclusion is not a perfect packing even when $n$ is divisible by $h$. Let $H$ be any 3-connected graph. Let $G$ be a graph obtained from two complete graphs on $\frac{n}{2} + 1$ vertices each where we identify 2 vertices from each part. Then each copy of $H$ must be completely contained in one of the parts. Therefore the only way there can be a perfect $H$-packing in $G$ is if $h$ divides $\frac{n}{2}$ or it divides both ($\frac{n}{2} - 1$ and $\frac{n}{2} + 1$).

### 7. Blow-up lemma

Chvátal, Rödl, Szemerédi, and Trotter’s theorem and Alon-Yuster theorem can be seen as preludes to the major breakthrough made by Komlós, Sarközy, and Szemerédi in 1997. It is famously known as the blow-up lemma.

Lemma naturally raises the following question: how large can $H$ be? In other words, can we relax the condition $m < (\delta - \varepsilon)n$? We definitely cannot hope for $m \geq (1 - \varepsilon)n$, since an $\varepsilon$-regular pair can have $\varepsilon n$ isolated vertices. Can we get close to this bound? Alon and Yuster’s theorem suggests that it might be possible.

**Definition 22.** A pair $(V_1, V_2)$ is $(\varepsilon, \delta)$-super-regular if it is $\varepsilon$-regular and $|N(x) \cap V_2| \geq \delta|V_2|$ for all $x \in V_1$ and $|N(x) \cap V_1| \geq \delta|V_1|$ for all $x \in V_2$.

The blow-up lemma asserts that we can find spanning subgraphs in super-regular pairs.

**Theorem 23.** (Blow-up lemma) For all $\delta, \Delta, r$, there exists $\varepsilon$ such that the following holds. Let $G$ be a graph with vertex partition $V = V_1 \cup V_2 \cup \cdots \cup V_r$ where $(V_i, V_j)$ is $(\varepsilon, \delta)$-super-regular for all distinct $i, j \in [r]$. Suppose that $H$ is an $r$-partite graph of maximum degree at most $\Delta$ with vertex partition $W = W_1 \cup \cdots \cup W_r$. If $|W_i| \leq |V_i|$ for all $i \in [r]$, then there exists a copy of $H$ in $G$ where $W_i$ gets mapped to $V_i$ for all $i \in [r]$.

Using this lemma, they settled several conjectures that we mentioned above. These include Bollobás’s conjecture on trees and Pósa-Seymour conjecture on power of Hamilton cycles and paths.

**Proof sketch.** Random Greedy Algorithm. Parameters are chosen so that $\varepsilon \ll \varepsilon' \ll \delta$. Furthermore $\varepsilon'$ (and thus $\varepsilon$) will be sufficiently small depending on $\Delta$ and $r$.

For simplicity, we will only discuss the case when $|V_1| = \cdots = |V_r| = m$.

**Step 1.** Preprocessing

For each $i \in [r]$, find $W'_i \subseteq W_i$ so that $|W'_i| = \varepsilon'|W_i|$ and the subgraph of $H$ induced on $W'_1 \cup \cdots \cup W'_r$ is 2-independent.
We will find the sets \( W'_1, \ldots, W'_r \) as above in order. Suppose that we found \( W'_1, \ldots, W'_{i-1} \). First remove from \( W_{i+1} \) all vertices that are at distance at most 2 from some vertex in \( W'_1 \cup \cdots \cup W'_{i-1} \). Since \( H \) has maximum degree at most \( \Delta \), this removes at most \( \Delta \) vertices from \( W_{i+1} \). Then construct an auxiliary graph \( \Gamma \) on the remaining vertices of \( W_{i+1} \) where two vertices are adjacent if and only if they are at distance at most 2 from each other in \( H \). Since \( H \) has maximum degree at most \( \Delta \), the graph \( \Gamma \) has maximum degree at most \( \Delta^2 \). Therefore it contains an independent set of size at least

\[
\frac{(1-\Delta^2)\varepsilon m}{\Delta^2+1} \geq \varepsilon' m.
\]

Let \( W'_{i+1} \) be an arbitrary independent set of size \( \varepsilon' m \) and repeat.

**Step 2. Random greedy embedding**

Define \( W''_i = W_i \setminus W'_i \) for each \( i \in [r] \). Let \( x_1, x_2, x_3, \ldots, x_T \) be an arbitrary enumeration of the vertices in \( W''_1 \cup \cdots \cup W''_r \). We will embed these vertices using an iterative algorithm where for \( t = 1, 2, \ldots, T \), we embed \( x_t \) at the \( t \)-th step. We use \( f \) to denote the partial embedding throughout this process. Suppose that we are at the \( t \)-th step. For each \( i \geq t \), define

\[
N_t(x_i) = N(x_i) \cap \{x_1, \ldots, x_{t-1}\}
\]

as the neighbors of \( x_i \) that have already been embedded. In the beginning of the \( t \)-th step, for each \( i \geq t \), we will be given as input a set \( U^{(t)}_i \) that is adjacent to \( f(u) \) for all \( u \in N_t(x_i) \) and has size at least

\[
|U^{(t)}_i| \geq (\delta - \varepsilon)|N_t(x_i)|m
\]

and satisfying

\[
|U^{(t)}_i \cap \{f(x_i) : i < t\}| \approx \alpha(\delta - \varepsilon)|N_t(x_i)|m,
\]

where \( \alpha \) is the proportion of vertices in \( W_j \) (the part that \( x_i \) belongs to) that has already been embedded.

Then \( f(x_t) \) will be chosen in \( U^{(t)}_i \) as in CRST proof given above. Thus for each vertex \( x_s \in N(x_t) \cap \{x_{t+1}, \ldots, x_T\} \), remove from \( U^{(t)}_i \), all vertices that have less than \( (\delta - \varepsilon)|U^{(t)}_s| \) neighbors in \( U^{(t)}_s \). Each vertex \( x_s \) will remove at most \( \varepsilon |U^{(t)}_i| \) vertices from \( U^{(t)}_i \), and therefore we are left with a set \( A_t \) of size at least \( (1 - \Delta \varepsilon)|U^{(t)}_i| \). Choose \( f(x_t) \) as a uniform random vertex in \( A_t \). This random choice will ensure (??).

**Step 3. Finalizing**

As an outcome of Step 2, we have a partial embedding \( f \) from \( \cup_{i \in [r]} W''_i \) to \( V(G) \). Let \( V'_i = V_i \setminus f(W''_i) \) be the subset of remaining vertices for each \( i \in [r] \). The final step is to embed \( W'_1 \) to \( V'_1 \) for each \( i \). By the independence property that we imposed on \( W'_1 \cup \cdots \cup W'_r \), it suffices to find an embedding \( W'_i \) to \( V'_i \) separately for each \( i \in [r] \).

For fixed \( i \in [r] \), construct an auxiliary bipartite graph \( H_i \) over two parts \( W'_i \cup V'_i \), where \( w \in W'_i \) is adjacent to \( v \in V'_i \) if and only if defining \( f(w) = v \) is consistent with the partial embedding. The random greedy algorithm also ensures that \( H_i \) satisfies Hall’s condition. Therefore we can find a perfect
matching between \( W'_i \) and \( V'_i \) in \( H_i \). Extend \( f \) to \( W'_i \) using this perfect matching. Repeat it for each \( i \in [r] \) to find an embedding with the desired properties.

8. Bandwidth Theorem

Komlós, Sarkőzy, and Szemerédi proved several conjectures regarding spanning subgraphs using the blow-up lemma such as Bollobás’s theorem on trees, Pósa-Seymour conjecture on power of Hamilton paths and cycles, and Alon-Yuster theorem (when the number of vertices is sufficiently large).

Motivated by such success, Bollobás and Komlós made a conjecture on the minimum degree threshold for existence of certain spanning subgraphs.

A bandwidth of an \( n \)-vertex graph \( G \) is the minimum integer \( b \) such that there exists a labelling of the vertices of \( G \) by \([n]\) such that \( |i-j| \leq b \) whenever \( \{i,j\} \) is an edge of \( G \). A typical example of a graph of small bandwidth is a power of a Hamilton path and cycle. The \( k \)-th power of a Hamilton cycle has bandwidth at most \( k \). A graph is known to have sublinear bandwidth if and only if it has no linear ‘expanding’ subgraph.

**Conjecture 24.** (Bollobás-Komlós) For every \( \Delta, r, \varepsilon \), there exists \( \beta \) such that the following holds. Every \( n \)-vertex graph \( G \) of minimum degree at least \((1 - \frac{1}{r} + \varepsilon)n\), contains all \( r \)-chromatic \( n \)-vertex graphs of maximum degree at most \( \Delta \) and bandwidth at most \( \beta n \) as subgraphs.

As seen before the minimum degree must be at least \((1 - \frac{1}{r})n\) (in fact even slightly large than this by the construction described after Alon and Yuster’s conjecture).

The conjecture is false if we completely remove the restriction on bandwidth. To see this, suppose that \( H \) is a \( d \)-regular \( r \)-chromatic \( n \)-vertex graph with the property that for every disjoint vertex subsets \( X, Y \) of sizes at least \(|X|, |Y| \geq \frac{n}{r} \), there exists an edge between \( X \) and \( Y \) (such graph exists if \( d \) is large enough). Let \( G \) a an \( n \)-vertex graph consisting of two complete graph on \((1 - \frac{1}{r})n \) vertices identified on \((1 - \frac{2}{r})n \) vertices. Note that \( H \) cannot be a subgraph of \( G \), since \( G \) has two disjoint vertex subsets of sizes \( \frac{n}{r} \) with no edge between them.

The conjecture above is powerful and ‘approximately’ contains all previously mentioned cases as subcases. The word approximately is used with two different meanings. First, the minimum degree is slightly weaker in that it is \((1 - \frac{1}{r} + \varepsilon)n\) instead of \((1 - \frac{1}{r})n\). Furthermore, it requires a minimum degree \( \frac{2}{3}n \) for a Hamilton cycle when \( n \) is odd, as opposed to \( \frac{1}{2}n \) established in Dirac’s theorem. Same weakness can be seen for Pósa-Seymour conjecture.

**Theorem 25.** (Böttcher-Schacht-Taraz 2009) For every \( \Delta, r, \varepsilon \), there exists \( \beta \) such that the following holds. Every \( n \)-vertex graph \( G \) of minimum degree at least \((1 - \frac{1}{r} + \varepsilon)n\), contains all \( r \)-chromatic \( n \)-vertex graphs \( H \) of maximum degree at most \( \Delta \) and bandwidth at most \( \beta n \) as subgraphs. Furthermore, the
conclusion holds for \((r + 1)\)-chromatic graphs \(H\) as long as one of the colors is ‘sparse’.

This theorem is referred to as the bandwidth theorem.

Recall that Counting lemma + Turán’s theorem implied Erdős-Stone theorem. Similarly, Blow-up lemma + Pósa-Seymour conjecture implies an almost-spanning version of bandwidth theorem (a spanning version requires more work).