1. Graph limits

Recall that for a fixed graph $H$, we defined $\text{Hom}(H, G)$ as the set of homomorphisms from $H$ to $G$. Define $h(H, G) = |\text{Hom}(H, G)|$ and

$$t(H, G) = \frac{h(H, G)}{|V(G)||V(H)|}.$$ 

Let $(G_n)_{n=1}^\infty$ be a sequence of graphs whose number of vertices tends to infinity. We say that $(G_n)_{n=1}^\infty$ is a convergent graph sequence if the sequence $(t(H, G_n))_{n=1}^\infty$ is a convergent sequence for every graph $H$. We say that $G_n$ converges to $G$ if

$$\lim_{n \to \infty} t(H, G_n) = t(H, G)$$

for all fixed graphs $H$.

There are sequences of graphs $(G_n)_{n=1}^\infty$ that are convergent but not converging to any graph $G$. Indeed, suppose that $G_n$ is a $n$-vertex random graph $G(n, \frac{\epsilon}{2})$. With probability $1 - o(1)$, for each fixed graph $H$ we have $t(H, G_n) \to 2^{-|E(H)|}$. However there is no graph $G$ for which $G_n$ converges to $G$. This raises the following question:

Is there a natural limit object of graph sequences?

Before further discussing this question, we extend the definition of graph homomorphism to weighted graphs. A weighted graph $G$ is a graph with a weight $\alpha(i)$ associated with each vertex and a weight $\beta(i, j)$ associated with each edge $ij$ ($G$ may have loops, but no multiple edges). We restrict our attention to positive real weights between 0 and 1. Note that a (non-weighted) graph $G$ may be considered as a weighted graph $G$ with $\alpha(i) = 1$ for all vertices, and $\beta(i, j) = 1$ if $ij$ is an edge, and $\beta(i, j) = 0$ if $ij$ is not an edge.

For a weighted graph $G$ and a non-weighted graph $H$, define

$$h(H, G) = \sum_{\phi: V(H) \to V(G)} \left( \prod_{i \in V(H)} \alpha(\phi(i)) \prod_{ij \in E(H)} \beta(\phi(i)\phi(j)) \right),$$

and define

$$t(H, G) = \frac{h(H, G)}{\left(\sum_{i \in V(G)} \alpha(i)\right)|V(H)|}.$$ 

This coincides with the definition above for non-weighted graphs. In most cases, for simplicity we will assume that $\sum_{i \in V(G)} \alpha(i) = 1$. 

We say that a sequence \((G_n)_{n=1}^{\infty}\) of weighted graphs is convergent if the sequence \((t(H, G_n))_{n=1}^{\infty}\) has a limit as \(n \to \infty\) for every non-weighted graph \(H\). We say that a sequence of weighted graphs converges to a finite weighted graph \(G\) if
\[
t(H, G_n) \to h(H, G)
\]
for every non-weighted graph \(H\).

**Example 1.** We obtain a richer family of limit objects by considering weighted graphs. For example in the random graph sequence example given above, the sequence converges to the weighted graph \(G\) on a single vertex \(\{1\}\) with a loop where \(\alpha(1) = 1\) and \(\beta(1, 1) = \frac{1}{2}\).

2. **Limit object**

Lovász and Szegedy showed that there is a natural limit object in the form of a symmetric measurable function \(W : [0, 1]^2 \to [0, 1]\) (where \(W\) is symmetric if \(W(x, y) = W(y, x)\)). The function \(W\) can also be viewed as an infinite weighted graph on the points of the unit interval.

For a graph \(H\) on vertex set \([k]\) and a symmetric function \(W : [0, 1]^2 \to [0, 1]\), define
\[
t(H, W) := \int_{[0,1]^n} \prod_{ij \in E(H)} W(x_i, x_j) dx_1 \cdots dx_n.
\]

For a weighted graph \(G\) on vertex set \([k]\), define \(W_G : [0, 1]^2 \to [0, 1]\) as follows (suppose that \(\sum_{i \in V(G)} \alpha(i) = 1\). For given \(x, y \in [0, 1]\), let \(a\) and \(b\) be vertices defined by
\[
\sum_{i=1}^{a-1} \alpha(i) \leq x < \sum_{i=1}^{a} \alpha(i) \quad \text{and} \quad \sum_{j=1}^{b-1} \alpha(j) \leq y < \sum_{j=1}^{b} \alpha(j).
\]
We let \(W_G(x, y) = \beta(a, b)\). Note that \(W_G\) is symmetric and
\[
t(H, G) = t(H, W_G)
\]
for all non-weighted graphs \(H\).

For a sequence \((G_n)_{n=1}^{\infty}\) of graphs and a symmetric measurable function \(W\), we say that \(G_n\) converges to \(W\) if
\[
\lim_{n \to \infty} t(H, G_n) = t(H, W)
\]
for all non-weighted graphs \(H\). Our first theorem asserts that all converging graph sequences converge to some symmetric function \(W\).

**Theorem 2.** Every converging graph sequence \((G_n)_{n=1}^{\infty}\) converges to some symmetric measurable function \(W : [0, 1]^2 \to [0, 1]\).

Note that \(W\) is not necessarily unique since for isomorphic graphs \(G\) and \(G'\), the corresponding symmetric functions \(W_G\) and \(W_{G'}\) satisfy
\[
t(H, W_G) = t(H, W_{G'})
\]
for all fixed graphs $H$, but we do not necessarily have $W_G = W_{G'}$.

Next theorem asserts that for every symmetric measurable function $W$, there exists a sequence of graphs $(G_n)_{n=1}^\infty$ that converge to $W$.

**Theorem 3.** For every symmetric measurable function $W : [0,1]^2 \to [0,1]$, there exists a sequence of graphs converging to $W$.

In this context, we call a symmetric measurable function $W : [0,1]^2 \to [0,1]$ a graphon. The theorems above shows that the space of graphons can be seen as the completion of the space finite graphs.

**Example 4.** The random graph sequence discussed above converges to the constant graphon $W(x,y) = \frac{1}{2}$.

**Example 5.** Turán graphs

**Example 6.** Let $G_{n,n}$ denote the bipartite graph on $2n$ vertices $A \cup B$ where $A, B$ are two disjoint copies of $[n]$. The vertices $i \in A, j \in B$ are adjacent if and only if $i \leq j$.

If $H$ is not a bipartite graph, then $t(H,G_{n,n}) = 0$. If $H$ is a bipartite graph, then define a partial order on $V(H) = V_1 \cup V_2$ where $v \leq w$ if $v \in V_1, w \in V_2$ are adjacent. All homomorphisms $f$ of $H$ to $G_{n,n}$ must either satisfy $(f(V_1) \subseteq A$ and $f(V_2) \subseteq B)$, or $(f(V_1) \subseteq B$ and $f(V_2) \subseteq A)$. By symmetry there are a same number of both types of homomorphisms. Note that $\frac{1}{2} h(H,G_{n,n})$ is the number of order preserving maps from $V(H)$ to $[n]$. Since

\[ t(H,G_{n,n}) = \frac{h(H,G_{n,n})}{(2n)^k} = \frac{1}{2^{k-1}} \frac{h(H,G_{n,n})}{n^k}, \]

the quantity $2^{k-1} t(H,G_{n,n})$ equals the probability that a uniform random map from $H$ to $G_{n,n}$ is order-preserving, which as $n$ tends to infinity approaches $\frac{i(H)}{k!}$, where $i(H)$ is the number of linear extensions of the partial order on $V(H)$.

**Example 7.** Consider the sequence of iterated blow-ups of $C_5$. The sequence converges to a fractal-like graphon.

### 3. Cut norm and graph distance

For an integrable function $U : [0,1]^2 \to \mathbb{R}$, define the rectangle norm by

\[ \|U\|_\square = \sup_{A \subseteq [0,1]} \left| \int_A \int_B U(x,y) dx dy \right|. \]

One can easily check that it indeed is a norm and that

\[ \|U\|_\square = \sup_{0 \leq f,g \leq 1} \int_{[0,1]^2} U(x,y) f(x) g(y) dx dy. \]

The following lemma relates the rectangle norm and homomorphism densities.
Lemma 8. Let $U, W : [0,1]^2 \rightarrow [0,1]$ be symmetric integrable functions. Then for every finite graph $H$,
\[
|t(H, U) - t(H, W)| \leq |E(H)| \cdot \|U - W\|_\square.
\]

Proof. let $E(H) = \{e_1, e_2, \cdots, e_m\}$. Then
\[
t(H, U) - t(H, W) = \int_{[0,1]^n} \left( \prod_{i=1}^t W(x(e_i)) - \prod_{i=1}^t U(x(e_i)) \right) dx,
\]
where for an edge $e = vw$ we write $x(e) = (v, w)$. Note that
\[
\prod_{i=1}^m W(x(e_i)) - \prod_{i=1}^m U(x(e_i))
\]
\[
= \prod_{i=1}^m W(x(e_i)) - \left( \prod_{i=1}^{m-1} W(x(e_i)) \right) U(x(e_m)) + \left( \prod_{i=1}^{m-1} W(x(e_i)) \right) U(x(e_m)) - \prod_{i=1}^m U(x(e_i))
\]
\[
= X_m(x) + \left( \prod_{i=1}^{m-1} W(x(e_i)) \right) U(x(e_m)) - \prod_{i=1}^m U(x(e_i))
\]
\[
= \cdots
\]
\[
= \sum_{i=1}^m X_i(x)
\]
where $X_i(x) = \left( \prod_{i=1}^{t-1} W(x(e_i)) \right) \left( \prod_{i=t+1}^m U(x(e_i)) \right) \cdot (W(x(e_t)) - U(x(e_t)))$.
If $e_t = ij$, then $X_i(x)$ can be written in the form $(W(x_i, x_j) - U(x_i, x_j))f(x_i, y)g(x_j, y)$ where $y$ is the length $n - 2$ vector obtained from $x$ by removing $x_i$ and $x_j$.

By (1) by integrating over $x_i$, $x_j$, and then over $y$, we have
\[
\| \int_{[0,1]^n} X_i(x) dx \| \leq \|U - W\|_\square.
\]
Therefore
\[
|t(H, U) - t(H, W)| \leq \sum_{t=1}^m \left| \int_{[0,1]^n} X_i(x) dx \right| \leq |E(H)| \cdot \|U - W\|_\square. \quad \square
\]

Note that for a sequence $(G_n)_{n=1}^\infty$, we have two notions of convergence to a graphon.
1. $t(H, G_n) \rightarrow t(H, W)$ for all fixed graphs $H$, and
2. $\|W_G - W\|_\square \rightarrow 0$.

By the lemma above 2 implies 1. On the other hand 1 does not imply 2 since when restricted to graphs, the distance is not invariant under relabeling. One can generalize the definition to take this into account (see Borgs, Chayes, Lovász, Sós, Vesztergombi for details). If properly modified, then convergent graph sequences can be equivalently defined in terms of the distance approaching zero.
A graphon $U : [0, 1]^2 \to [0, 1]$ is a $k$-stepfunction if there exists a partition $[0, 1] = S_1 \cup \cdots \cup S_k$ such that $U$ is constant on every set $S_i \times S_j$.

**Theorem 9.** (Weak regularity lemma) For every $\varepsilon > 0$, there exists an integer $k \leq 2^{O(1/\varepsilon^2)}$ such that for every graphon $W$, there exists a $k$-stepfunction graphon $U$ for which $\|W - U\|_\Box \leq \varepsilon$.

Lemma 8 implies that the number of copies of a graph $H$ in another graph $G$ can be 'easily' approximated, given a weak regular partition.

### 4. INDUCED AND INJECTIVE HOMOMORPHISMS

**Definition 10.** For two graphs $H$ and $G$, a strong homomorphism is a map $f : V(H) \to V(G)$ such that for a pair of vertices $v, w \in V(H)$, the pair $\{f(v), f(w)\}$ forms an edge of $G$ if and only if $\{v, w\}$ is an edge of $H$. Let $\text{Som}(H, G)$ be the set of strong homomorphisms from $H$ to $G$.

**Remark.** $\text{Som}(H, G)$ is not a standard notation.

Define $\text{Som}(H, G)$ as the set of strong homomorphisms from $H$ to $G$ and

$$s(H, G) = \frac{|\text{Som}(H, G)|}{|V(G)||V(H)|}.$$

If $H$ is a $k$-vertex graph, then

$$t(H, G) = \sum_{H' \subseteq K_k} s(H', G).$$

Also by the inclusion-exclusion principle,

$$s(H, G) = \sum_{H' \subseteq H \atop |V(H')|=k} (-1)^{|E(H') \setminus E(H)|} t(H', G).$$

Therefore for all graphs $G$, the sequence $\{t(H, G)\}_H$ is completely determined by $\{s(H, G)\}_H$ and vice versa.

If $G$ is a weighted graph whose vertex weights are $\alpha$ and edge weights are $\beta$, then it is natural to define

$$s(H, G) = \sum_{\phi : V(H) \to V(G)} \left( \prod_{i \in V(H)} \alpha(\phi(i)) \prod_{ij \in E(H)} \beta(\phi(i)\phi(j)) \prod_{ij \notin E(H)} (1 - \beta(\phi(i)\phi(j))) \right),$$

and similarly for a graphon $W$ (when $\sum_{i \in V(H)} \alpha(i) = 1$),

$$s(H, W) = \int_{[0,1]^n} \prod_{ij \in E(H)} W(x_i, x_j) \prod_{ij \notin E(H)} (1 - W(x_i, x_j)).$$

Lovász and Szegedy studied various theoretical aspects of graphons. Razborov found a similar framework that can be used in attacking Turán-type problems. Note that for Turán-type problems, we are interested in injective
homomorphisms. For two graphs $H$ and $G$, define $\text{Hom}^0(H, G)$ as the set of injective homomorphisms from $H$ to $G$ and

$$t_0(H, G) = \frac{|\text{Hom}^0(H, G)|}{|V(G)|(|V(G)| - 1) \cdots (|V(G)| - |V(H)| + 1)}.$$ 

If $(G_n)_{n=1}^{\infty}$ is a sequence of graphs whose number of vertices tends to infinity as $n$ goes to infinity, then for each fixed graph $H$,

$$t(H, G_n) \rightarrow t_H \iff t_0(H, G_n) \rightarrow t_H.$$ 

Therefore asymptotically, there is no difference between homomorphisms and injective homomorphisms. Similarly define $\text{S_{om}}^0(H, G)$ and $s_0(H, G)$.

5. Flag Algebra

5.1. Vector Space. Let $W$ be the space of graphons. Throughout this section, for each graph $H$, we fix a labelling and use $H$ to denote the functional $W \rightarrow \mathbb{R}$ defined as $H(W) = \frac{|V(H)|!}{|\text{Aut}(H)|} \cdot s(H, W)$ for all $W \in W$. We divide the right-hand-side by $|\text{Aut}(H)|$ to account for different labellings. For a graph $G$, define $H(G) = H(W_G)$.

For graphs $H$ and $G$, define $p(H, G)$ as the probability that a uniform random $|V(H)|$-tuple of vertices of $G$ forms an induced copy of $H$. Note that $p(H, G) = \frac{|V(H)|!}{|\text{Aut}(H)|} \cdot s_0(H, G)$. Further note that if $(G_n)_{n=1}^{\infty}$ is a sequence of graphs with increasing number of vertices converging to $W \in W$, then $p(H, G_n) \rightarrow H(W)$. However note that $p(H, G_n) \neq H(G_n)$.

Consider the vector space $F$ consisting of finite formal sums of graphs with coefficient in $\mathbb{R}$. Naturally we define as a functional,

$$\left( \sum_{i=1}^{k} c_i H_i \right)(W) = \sum_{i=1}^{k} c_i \cdot H_i(W).$$

Example 11. Let $\overline{K}_3$ be the empty graph on three vertices. Then $K_3 + \overline{K}_3$ can be seen as an operator which when applied to a graph $G$ computes the density of $K_3$ plus the density of $\overline{K}_3$ in $G$. As we have seen in the problem set, $(K_3 + \overline{K}_3)(G) \geq \frac{1}{4}$ holds for all graphs $G$.

For two elements $F_1, F_2 \in F$, we write $F_1 \geq F_2$ if $F_1(W) \geq F_2(W)$ holds for all graphons $W \in W$. By Theorem 3, we have $F_1 \geq F_2$ if and only if $F_1(G) \geq F_2(G)$ holds for all graphs $G$. Let 1 be the graph with one vertex and (note that $1(W) = 1$ for all graphons $W$).

Example 12. From the discussion above, we see that

$$K_3 + \overline{K}_3 \geq \frac{1}{4}.$$ 

There exist non-trivial relations between elements in $F$. For example we have $1 = K_2 + \overline{K}_2$. Similarly if $F_k$ is the set of all graphs on $k$ vertices (having one graph for each isomorphism class), then $1 = \sum_{H \in F_k} H$. 
Proposition 13. For each graph $H$ and $k \geq |V(H)|$, the following holds:

$$H = \sum_{F \in \mathcal{F}_k} p(H, F) \cdot F.$$ 

Proof. By Theorem 3, it suffices to prove that

$$\frac{|V(H)|!}{|\text{Aut}(H)|} s(H, G) = (1 + o_n(1)) \sum_{F \in \mathcal{F}_k} p(H, F) \cdot \frac{|V(H)|!}{|\text{Aut}(F)|} s(F, G).$$

holds for all sufficiently large graphs $G$. Hence it suffices to prove that

$$\frac{|V(H)|!}{|\text{Aut}(H)|} s_0(H, G) = \sum_{F \in \mathcal{F}_k} p(H, F) \cdot \frac{|V(F)|!}{|\text{Aut}(F)|} s_0(F, G)$$

or equivalently $p(H, G) = \sum_{F \in \mathcal{F}_k} p(H, F)p(F, G)$. The left-hand-side equals the probability that a uniform random $|V(H)|$-tuple of vertices in $G$ forms an induced copy of $H$. The right-hand-side computes the same probability by first considering a uniform random $k$-tuple of vertices of $G$. □

Note that for a fixed graph $H$, the extremal number of $H$ is the solution to the following optimization problem

Maximize $K_2(W)$

Subject to $W \in \mathcal{W}$ and $H'(W) = 0 \ \forall H' \supseteq H$.

5.2. Graph Algebra. $\mathcal{F}$ in fact has an algebra structure. For two graphs $H_1$ and $H_2$, it is natural to define $H_1 \cdot H_2$ as the element in $\mathcal{F}$ such that $(H_1 \cdot H_2)(W) = H_1(W) \cdot H_2(W)$ for all $W \in \mathcal{W}$. Does there exist such element in $\mathcal{F}$?

For graphs $H_1, H_2,$ and $G$ satisfying $k_i = |V(H_i)|$ for $i = 1, 2$ and $|V(G)| \geq k_1 + k_2$, define $p(H_1, H_2; G)$ as the probability that a uniform random $(k_1 + k_2)$-tuple of vertices $v_1, v_2, \ldots, v_{k_1+k_2}$ in $G$ satisfies the following: the subgraph of $G$ induced on $\{v_1, \ldots, v_{k_1}\}$ forms an induced copy of $H_1$ and the subgraph of $G$ induced on $\{v_{k_1+1}, \ldots, v_{k_1+k_2}\}$ forms an induced copy of $H_2$.

Proposition 14. For all $k \geq k_1 + k_2$,

$$p(H_1, G) \cdot p(H_2, G) = (1 + o_n(1)) \sum_{F \in \mathcal{F}_k} p(H_1, H_2; F) \cdot p(F, G),$$

Proof. Similar to Proposition 13. We omit the proof. □

Since $p(H, G) = (1 + o_n(1))H(G)$ we can naturally define

$$H_1 \cdot H_2 = \sum_{F \in \mathcal{F}_{k_1+k_2}} p(H_1, H_2; F) \cdot F.$$ 

Example 15. Note that $1 \cdot 1 = K_2 + \overline{K_2} = 1$. 

Example 16. There are 6 triangle free graphs on 4 vertices. Denote them as $F_0, F_1, F_2^{(0)}, F_2^{(1)}, F_3^{(0)}, F_3^{(1)}, F_4$ where for each $i$ the subscript indicates the number of edges. ($F_2^{(0)}$ is a matching and $F_2^{(1)}$ is $K_{1, 2}$ with an isolated vertex, $F_3^{(0)}$ is a star and $F_3^{(1)}$ is a path).

When restricted to the subspace of $W$ satisfying $K_3(W) = 0$, we have

$$K_2 \cdot K_2 = \frac{1}{4} F_2^{(0)} + \frac{1}{3} F_3^{(1)} + \frac{2}{3} F_4.$$  

Similarly $K_2 \cdot K_2 = \frac{1}{6} F_1 + \frac{1}{3} F_2^{(1)} + \frac{1}{3} F_3^{(0)} + \frac{1}{6} F_3^{(1)}$ and $K_2 \cdot K_2 = F_0 + \frac{2}{3} F_1 + \frac{2}{3} F_2^{(0)} + \frac{1}{3} F_2^{(1)} + \frac{1}{3} F_3^{(1)} + \frac{1}{3} F_4$. Note that

$$K_2 \cdot K_2 + 2K_2 \cdot K_2 + K_2 \cdot K_2 = F_0 + F_1 + F_2^{(0)} + F_2^{(1)} + F_3^{(0)} + F_3^{(1)} + F_4.$$  

Recall that the extremal number of $K_3$ is the solution to the following optimization problem

\[
\begin{align*}
\text{Maximize} & \quad K_2(W) \\
\text{Subject to} & \quad W \in \mathcal{W} \quad \text{and} \quad K_3(W) = 0.
\end{align*}
\]

One can try to solve this problem (or find an approximate solution to this problem) by maximizing the objective function over a larger space (thus relaxing the problem).

Restrict the attention to graphons $W \in \mathcal{W}$ satisfying $K_3(W) = 0$. Consider the following $2 \times 2$ symmetric matrix whose rows and columns are labelled by $K_2$ and $K_2$ and entries are in $F_4$

$$\begin{pmatrix}
K_2 \cdot K_2 & K_2 \cdot K_2 \\
K_2 \cdot K_2 & K_2 \cdot K_2
\end{pmatrix} = \begin{pmatrix}
\frac{1}{3} F_2^{(0)} + \frac{1}{3} F_2^{(1)} + \frac{2}{3} F_4 \quad \frac{1}{3} F_1 + \frac{1}{3} F_2^{(1)} + \frac{1}{3} F_3^{(0)} + \frac{1}{3} F_2^{(1)} \\
\frac{1}{3} F_1 + \frac{1}{3} F_2^{(1)} + \frac{1}{3} F_3^{(0)} + \frac{1}{3} F_3^{(1)} \quad F_0 + \frac{2}{3} F_1 + \frac{2}{3} F_2^{(0)} + \frac{1}{3} F_2^{(1)} + \frac{1}{3} F_3^{(1)} + \frac{1}{3} F_4
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} F_0 + \begin{pmatrix}
0 & \frac{1}{3} \\
\frac{1}{3} & 0
\end{pmatrix} F_2^{(0)} + \begin{pmatrix}
0 & \frac{1}{3} \\
\frac{1}{3} & 0
\end{pmatrix} F_3^{(0)} + \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{pmatrix} F_3^{(1)} + \begin{pmatrix}
\frac{2}{3} & 0 \\
0 & \frac{1}{3}
\end{pmatrix} F_4.
$$

The matrix above is positive semidefinite, i.e., for all $a$ and $b$

$$(a \quad b) \begin{pmatrix}
K_2 \cdot K_2 & K_2 \cdot K_2 \\
K_2 \cdot K_2 & K_2 \cdot K_2
\end{pmatrix} \begin{pmatrix}
a \\
b
\end{pmatrix} = (aK_2 + bK_2)^2 \geq 0,$$

where the last inequality holds since $(aK_2(W) + bK_2(W))^2 \geq 0$ for all $W \in \mathcal{W}$. Furthermore note that

$$K_2 = \frac{1}{6} F_1 + \frac{2}{6} (F_2^{(0)} + F_2^{(1)}) + \frac{3}{6} (F_3^{(0)} + F_3^{(1)}) + \frac{4}{6} F_4.$$
Hence the following optimization problem relaxes the optimization problem computing the extremal number of $K_3$:

Maximize $\frac{1}{6}x_1 + \frac{2}{6}(x_2^{(0)} + x_2^{(1)}) + \frac{3}{6}(x_3^{(0)} + x_3^{(1)}) + \frac{4}{6}x_4$

Subject to

\[
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x_0 + \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} x_1 + \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix} x_2^{(0)} + \\
\begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} x_2^{(1)} + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix} x_3^{(0)} + \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix} x_3^{(1)} + \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} x_4 \geq 0
\]

$x_0 + x_1 + x_2^{(0)} + x_2^{(1)} + x_3 + x_4 \leq 1$.

The above is a semi-definite programming problem and can be (approximately) solved in polynomial time.

5.3. Flag Algebra. For a graph $\sigma$, a $\sigma$-flag is a pair $(M, \phi)$ where $\phi$ is a homomorphism from $F$ to $M$. For simplicity, we sometimes omit $\phi$ and say that $M$ is a $\sigma$-flag.

Two $\sigma$-flags $(F_1, \phi_1)$ and $(F_2, \phi_2)$ are isomorphic if there exists an isomorphism $f : V(F_1) \to V(F_2)$ such that $f \circ \phi_1 = \phi_2$. For each $k \geq 1$, let $\mathcal{F}_k^\sigma$ be the set of (non-isomorphic) $\sigma$-flags on $k$ vertices. The element $1_\sigma \in \mathcal{F}_0^\sigma$ is $(\sigma, \phi)$ where $\phi$ is the identity map.

Example 17. Let $\sigma$ be an empty graph. Then a $\sigma$-flag is a graph.

Example 18. Let $\sigma$ be a single vertex graph. Then a $\sigma$-flag is a graph with a ‘distinguished’ vertex. For example, there are two distinct $\sigma$-flags on two vertices. Let $\rho$ be the unique $\sigma$-flag on two vertices with an edge, and let $\overline{\rho}$ be the unique $\sigma$-flag on two vertices with no edge. As in graph algebra, one can check that $1_\sigma = \rho + \overline{\rho}$.

Let $\mathcal{F}_k^\sigma$ be the set of finite formal sums of $\sigma$-flags with coefficient in $\mathbb{R}$. Suppose that $|V(\sigma)| = k$. For two $\sigma$-flags $H_1$ and $H_2$ satisfying $|V(H_1)| = k_1$ and $|V(H_2)| = k_2$, define

$$H_1 \cdot H_2 = \sum_{F \in \mathcal{F}_{k_1+k_2-k}} p(H_1, H_2; F) F,$$

where $p(H_1, H_2; F)$ is defined as follows. Let $F = (F, \phi)$ and let $F' \subseteq F$ be the image of $\phi$. For a set $X \subseteq V(F) \setminus V(F')$ of size $k_1 - k$ chosen uniformly at random, $p(H_1, H_2; F)$ is the probability that $X \cup V(F')$ forms a copy of $H_1$ as a $\sigma$-flag (where $V(F')$ is the image of $\sigma$), and $X \cup (V(F) \setminus V(F'))$ forms a copy of $H_2$ as a $\sigma$-flag.

Example 19. Then $\rho \cdot \overline{\rho} = \frac{1}{2} F_1^\sigma + \frac{1}{2} F_2^\sigma$ where $F_1^\sigma$ and $F_2^\sigma$ are certain three vertex $\sigma$-flags whose underlying graph has one edge and two edges, respectively.

Definition 20. For a type $\sigma$, define the downward operator $[] : \mathcal{F}_k^\sigma \to \mathcal{F}$ as $[(F, \phi)] = q_\sigma(F, \phi) F$, where $q_\sigma(F, \phi)$ is the probability that for an injective map $\psi : V(\sigma) \to F$, the $\sigma$-flag $(F, \psi)$ is isomorphic to $(F, \phi)$. 

With a slight technical twist, a $\sigma$-flag can also be interpreted as a functional on the space of graphons where $(F, \phi) \in \mathcal{F}_k^\sigma$ is a functional

$$(F, \phi) : W \times [0, 1]^{n-|\sigma|} \to \mathbb{R}$$

defined as

$$(F, \phi)(W, x_1, x_2, \cdots, x_k) = \frac{(k - |\sigma|)!}{|\text{Aut}(F, \phi)|} \int_{[0,1]^{n-|\sigma|}} \prod_{ij \in E(F)} W(x_i, x_j) \prod_{ij \notin E(F)} (1 - W(x_i, x_j)) dx$$

where $x = (x_{|\sigma|+1}, \cdots, x_n) \in [0, 1]^{n-|\sigma|}$. Furthermore $[]$ can be defined as

$$[[F, \phi]](W) = \int_{[0,1]^{n-|\sigma|}} (F, \phi)(W, y_1, \cdots, y_k) dy$$

where $y = (y_1, \cdots, y_k) \in [0, 1]^{n-|\sigma|}$. To see this note that

$$\int_{[0,1]^{n-|\sigma|}} (F, \phi)(W, y_1, \cdots, y_k) dy = \frac{(k - |\sigma|)!}{|\text{Aut}(F, \phi)|} \int_{x \in [0,1]^n} \prod_{ij \in E(F)} W(x_i, x_j) \prod_{ij \notin E(F)} (1 - W(x_i, x_j)) dx$$

$$= \frac{(k - |\sigma|)! |\text{Aut}(F, \phi)|}{k!} F(W) = q_{\sigma}(F, \phi) \cdot F(W).$$

The interpretation as above immediately implies the following Cauchy-Schwartz inequality.

**Proposition 21.** For a formal sum $F \in \mathcal{F}^\sigma$,

$$[[F^2]] \geq [[F]]^2.$$

The following example is known as Goodman’s bound.

**Example 22.** Recall that $\rho \cdot \overline{\rho} = \frac{1}{2} F_1^\sigma + \frac{1}{2} F_2^\sigma$. One can easily check that $[[F_1^\sigma]] = \frac{2}{3} F_1$ and $[[F_2^\sigma]] = \frac{2}{3} F_2$, where $F_1$ and $F_2$ are the three vertex graphs with one edge and two edges, respectively. Since $\rho + \overline{\rho} = 1$, we see that $\rho \cdot \overline{\rho} = \rho(1 - \rho)$ and therefore

$$\frac{1}{3} (F_1 + F_2) = [[\rho \cdot \overline{\rho}]] = [[\rho - \rho^2]] = [[\rho]] - [[\rho^2]] \leq [\rho] - [\rho]^2 = K_2 - K_2^2 \leq \frac{1}{4}. $$

Since $K_3 + F_1 + F_2 + K_3 = 1$, we see that $K_3 + K_3 \geq 1 - (F_1 + F_2) \geq 1$. 
