LECTURE 4. REGULARITY LEMMA II

1. Stability : Approximate to exact

By the Erdős-Stone theorem, we know that \( \text{ex}(n, C_{2k+1}) = (\frac{1}{4} - o(1))n^2 \) for all \( k \geq 1 \). We will prove the following strengthening of this result using the regularity lemma.

**Theorem 1.** For all \( k \geq 1 \), if \( n \) is sufficiently large then \( \text{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{T} \right\rfloor \).

For a partition \( \Pi = V_0 \cup V_1 \cup \cdots \cup V_k \) of a graph \( G \), we say that an edge \( e \) of \( G \) is relevant to the \((\epsilon, \delta)\)-reduced graph \( R(\Pi) \) if the two endpoints of \( e \) intersect distinct pairs \((V_i, V_j)\) that form an \( \epsilon \)-regular pair of density at least \( \delta \).

**Lemma 2.** If \( \Pi = V_0 \cup V_1 \cup \cdots \cup V_k \) is an \( \epsilon \)-regular partition of \( G \) with \( k \geq \frac{1}{\epsilon} \), then all but at most \((3\epsilon + \frac{\delta}{2})n^2\) edges are relevant to the \((\epsilon, \delta)\)-reduced graph \( R(\Pi) \).

**Proof.** The following are the non-relevant edges:

(i) edges incident to \( V_0 \) (at most \( \epsilon n^2 \) edges),
(ii) edges inside \( V_i \) for all \( i \) (at most \( k \cdot (\frac{n}{k})^2 = \epsilon n^2 \) edges),
(iii) edges between irregular pairs (at most \( \epsilon k \cdot (\frac{n}{k})^2 = \epsilon n^2 \) edges).
(iv) edges between regular pairs of density at most \( \delta \) (at most \( \delta (\frac{n}{k})^2 (\frac{k}{2}) < \frac{\delta}{2} n^2 \) edges).

In order to prove the theorem above, we first prove the following stability theorem.

**Theorem 3.** For every \( k \geq 1 \) and \( \epsilon \), there exist \( \delta \) and \( n_0 \) such that if a graph \( G \) on \( n \) vertices is \( C_{2k+1} \)-free and has at least \( (\frac{1}{4} - \delta)n^2 \) edges, then one can remove at most \( \epsilon n^2 \) edges to make it bipartite.

**Proof.** Recall that we previously proved the \( k = 1 \) case of this theorem. Suppose that \( k \geq 2 \). Given \( \epsilon \), let \( \delta \) be small enough so that every triangle-free graph of at least \( (\frac{1}{4} - 2\delta)n^2 \) edges can be made bipartite by removing at most \( \frac{\epsilon}{2} n^2 \) edges.

Let \( G \) be a \( C_{2k+1} \)-free graph on \( n \) vertices with at least \( (\frac{1}{4} - \delta)n^2 \) edges. Let \( \epsilon' \) be sufficiently small depending on \( \delta \) and \( \epsilon \). Take an \( \epsilon' \)-regular partition \( \Pi = V_0 \cup V_1 \cup \cdots \cup V_k \) of \( G \) where \( \frac{1}{\epsilon} \leq k \leq T \). Note that the \((\epsilon', \frac{\delta}{2})\)-reduced graph \( R(\Pi) \) is triangle-free as otherwise we can find a \( C_{2k+1} \) in \( G \).
By Lemma 2, there are at most \((3\varepsilon' + \frac{\varepsilon}{2})n^2 \leq \frac{\varepsilon}{2}n^2\) edges not relevant to \(R(\Pi)\). Thus \(R(\Pi)\) is a \(k\)-vertex graph with at least
\[
\left(\frac{1}{4} - \delta\right)n^2 - \frac{\varepsilon}{2}n^2 \geq \left(\frac{1}{4} - 2\delta\right)n^2
\]
edges. Hence \(R(\Pi)\) can be made bipartite by removing at most \(\frac{\varepsilon}{2}k^2\) edges.

Therefore by removing all the non-relevant edges and at most \(\left(\frac{n}{k}\right)^2 \cdot \frac{\varepsilon}{2}k^2 = \frac{\varepsilon}{2}n^2\) more edges, we can make \(G\) bipartite. Thus \(G\) can be made bipartite by removing at most \(\varepsilon n^2\) edges.

We now prove Theorem 1.

**Proof of Theorem 1.** The complete bipartite graph shows that \(ex(n, C_{2k+1}) \geq \left\lceil \frac{n^2}{4}\right\rceil\). Thus it suffices to prove the other direction of inequality for sufficiently large \(n\). We already proved the theorem for \(k = 1\) and hence we may assume that \(k \geq 2\).

For convenience, assume that \(n\) is even, and suppose that \(G\) is an \(n\)-vertex \(C_{2k+1}\)-free graph with at least \(\frac{n^2}{4} + m\) edges for some \(m > 0\). Let \(A \cup B\) be a partition of \(V(G)\) with the maximum number of crossing edges and let \(H\) be the bipartite graph induced on the partition \(A \cup B\). Note that \(e_H(A, B) \geq \left(\frac{1}{4} - \varepsilon\right)n^2\). If \(|A| = (\frac{1}{2} + \alpha)n\) and \(|B| = (\frac{1}{2} - \alpha)n\), then
\[
|A| \cdot |B| = \left(\frac{1}{4} - \alpha^2\right) \geq e_H(A, B) > \left(\frac{1}{4} - \varepsilon\right)n^2,
\]
and therefore \(\alpha \leq \sqrt{\varepsilon}\).

Let \(\{x, y\} \in E(G)\) be an edges whose both endpoints are in \(A\). Suppose that both \(x\) and \(y\) have at least \(\sqrt{\varepsilon}n\) neighbors in \(B\). Let \(X_1 = \{x\}\) and \(X_{2k+1} = \{y\}\). Define \(X_2 = N_H(X_1)\) and \(X_{2k} = N_H(X_{2k+1})\) where for a set \(X\), \(N_H(X)\) is the set of vertices incident to some vertex in \(X\) in the graph \(H\). For \(i = 3, 4, \cdots, 2k - 1\), define \(X_{i+1} = N_H(X_i)\). We claim that \(|X_i| \geq 4\varepsilon n\) for all \(i\). The claim is true for \(X_2\). Now suppose that the claim is true for \(X_i\) and not for \(X_{i+1}\). If \(X_i \subseteq A\), then \(X_{i+1} \subseteq B\) and
\[
e_H(X_i, B \setminus X_{i+1}) = 0.
\]
Therefore
\[
e_H(A, B) \leq |A||B| - |X_i| \cdot |B \setminus X_{i+1}|
\leq \frac{1}{4}n^2 - \sqrt{\varepsilon}n \cdot (|B| - \sqrt{\varepsilon}n) \leq \left(\frac{1}{4} - \varepsilon\right)n^2
\]
which contradicts (1). Similar analysis works when \(X_i \subseteq B\).

Hence \(|X_i| \geq \sqrt{\varepsilon}n\) for all \(i = 2, 3, \cdots, 2k\). Suppose that there are no edges between \(X_{2k-1}\) and \(X_{2k}\). Then
\[
e_H(A, B) \leq |A||B| - |X_{2k-1}\cdot X_{2k}| \leq \left(\frac{1}{4} - \varepsilon\right)n^2
\]
Thus we see that 

\[ \sqrt{\varepsilon n} \]

and contradicts (1). Hence there exists a cycle of length \(2k + 1\) containing both \(x\) and \(y\). Therefore for each edge \(\{x, y\} \in E(G)\) whose both endpoints are in \(A\), either \(d_H(x) < \sqrt{\varepsilon n}\) or \(d_H(y) < \sqrt{\varepsilon n}\).

Hence we can find a set \(A' \subseteq A\) that intersects all edges in \(A\) and \(d_H(a) < \sqrt{\varepsilon n}\) for all \(a \in A'\). Since \(A \cup B\) is a MAXCUT of \(G\), for each vertex \(a \in A\), we have \(\sqrt{\varepsilon n} \geq |N(a) \cap B| \geq |N(a) \cap A|\) (otherwise we can move \(a\) to \(B\) to obtain a larger cut). Hence 

\[ e_G(A) \leq |A'| \cdot \sqrt{\varepsilon n} \]

Also since each vertex \(a \in A'\) satisfies \(d_H(a) < \sqrt{\varepsilon n}\), we see that 

\[ e_G(A, B) \leq |A||B| - (|B| - \sqrt{\varepsilon n}) \cdot |A'| \leq |A||B| - \frac{n}{4}|A'|. \]

Similarly, we can find a set \(B' \subseteq B\) that intersects all edges in \(B\) and \(d_H(b) < \sqrt{\varepsilon n}\) for all \(b \in B'\). Moreover \(e_G(B) \leq |B'| \sqrt{\varepsilon n}\) and \(e_G(A, B) \leq |A||B| - \frac{n}{4}|B'|\).

Hence 

\[ |E(G)| \leq e_G(A) + e_G(B) + e_G(A, B) \leq (|A'| + |B'|) \cdot \sqrt{\varepsilon n} + |A||B| - \frac{1}{2} \cdot \frac{n}{4} (|A'| + |B'|) \leq |A||B|. \]

Thus we see that \(|A'| + |B'| = 0\) and \(e(G) \leq \frac{n^2}{4}\). \(\square\)

A graph \(G\) is color critical if for every edge \(e \in E(G)\), the graph \(G \setminus e\) (the graph obtained by removing \(e\)) satisfies \(\chi(G \setminus e) < \chi(G)\). Note that all odd cycles are color critical. The following theorem extends Theorem 3 to color critical graphs.

**Theorem 4.** (Simonovits) If \(H\) is a color critical graph, then for all sufficiently large \(n\),

\[ ex(n, H) = e(T_{n, \chi(H) - 1}), \]

where \(T_{n, \chi(H) - 1}\) is the \((\chi(H) - 1)\)-partite Turán graph on \(n\) vertices.

**Proof sketch.** Define \(r = \chi(H) - 1\). For simplicity, assume that \(n\) is divisible by \(r\). Let \(G\) be a graph on \(n\) vertices with at least \((1 - \frac{1}{2})\left(\frac{n}{2}\right)\) edges. Let \(V = V_1 \cup V_2 \cup \cdots \cup V_r\) be a \(r\)-partition of \(G\) which maximizes the number of edges across the partition. An extension of Theorem 3 can be used to show that the number of edges across the partition is at least \((1 - \frac{1}{2} - \varepsilon)\left(\frac{n}{2}\right)\) edges.

**Claim 1.** If \(xy\) is an edge whose both endpoints are in \(V_1\), then there exists an index \(i > 1\) such that \(x\) or \(y\) has at most \(\varepsilon' n\) neighbors in \(V_i\).

We omit the proof of the claim. They key idea is that for all subsets \(X_1 \subseteq V_i\) and \(X_2 \subseteq V_j\) of sizes \(|X_i| \geq \varepsilon' n\) and \(|X_j| \geq \varepsilon' n\), the number of edges between \(X_i\) and \(X_j\) is at least \(|X_i||X_j| - \varepsilon n^2\). Claim 1 implies the following claim.
**Claim 2.** For each index $i$, there exists a set $W_i \subseteq V_i$ that (i) intersects all edges of $G$ inside $V_i$, (ii) $e_G(V_i) \leq |W_i| \cdot \varepsilon'n$, and (iii) the number of edges across the partition is at most $(1 - \frac{1}{r})\binom{n}{2} - \frac{n}{2r}|W_i|$.

It easily follows that $W_i = \emptyset$ for all $i$. □

2. **Weak regularity lemma**

Recall that the bound on the number of parts in the regularity lemma is rather poor. In this section, we study variants of the regularity lemma where the definition of regularity is weaker but the bound on the number of parts is dramatically better.

2.1. **Frieze and Kannan.** The weak regularity lemma of Frieze and Kannan is based on the following definition of regularity.

**Definition 5.** Let $G$ be a graph on $n$ vertices. A partition $\Pi = V_0 \cup V_1 \cup \cdots \cup V_k$ is weak $\varepsilon$-regular if $|V_0| \leq \varepsilon n$, $\Pi$ is an equitable partition, and for every pair of disjoint subsets $X, Y$ of sizes $|X| \geq \varepsilon n$ and $|Y| \geq \varepsilon n$,

$$ \left| e(X,Y) - \sum_{i \neq j} d(V_i,V_j) \cdot |X \cap V_i| \cdot |Y \cap V_j| \right| \leq \varepsilon n^2. $$

The quantity above measures the difference between the number of edges between $X$ and $Y$ and the expected number of edges based on the densities across the pairs of parts of the partition. The following theorem is the weak regularity lemma of Frieze and Kannan.

**Theorem 6.** For all $t$ and $\varepsilon$, there exists $T$ and $n_0$ such that the following holds for all $n \geq n_0$. Every $n$-vertex graph admits a weak $\varepsilon$-regular partition into $k$ parts for some $t \leq k \leq T$ where $T \leq 2^{-\Omega(\varepsilon^2)}$.

The bound $2^{-\Omega(\varepsilon^2)}$ is best possible. The proof of Theorem 6 is similar to the proof of the original regularity lemma. The key difference is that the number of parts do not increase exponentially. Following is the main ingredient in the proof of the weak regularity lemma.

**Lemma 7.** Let $\Pi = (V_i)_{i=0}^k$ be an equitable partition satisfying $|V_0| \leq \varepsilon n$ that is not weak $\varepsilon$-regular. Then there exists a partition $\Pi'$ such that

$$ d_2(\Pi') \geq d_2(\Pi) + \varepsilon^2. $$

Moreover $|\Pi'| \leq 3k$.

The main gain compared to the original regularity lemma comes from the fact that we no longer need to take the common refinement of partitions. Frieze and Kannan developed the weak regularity lemma with algorithmic applications in mind. In fact for those applications we need the following slightly different form of the lemma.
Theorem 8. For every positive real \( \varepsilon \) and graph \( G \), there exists a family of \( k \) triples \((S_i, T_i, d_i)\) where \( S_i, T_i \) are disjoint subsets of vertices and \( d_i \) is a real number, such that for every disjoint sets \( X, Y \subseteq V(G) \),

\[
|e(X, Y) - \sum_{i=1}^{k} d_i |S_i \cap X||T_i \cap Y| | \leq \varepsilon n^2,
\]

where \( k \leq 1/\varepsilon^2 \).

Note that \( d_i \) are not necessarily positive nor bounded. The best way to prove the theorem above is to go through matrices.

I will discuss one application. The MAXCUT problem takes a graph \( G \) as input and asks to determine the largest bipartite subgraph of \( G \). Let \( \text{OPT} \) be the number of edges in the largest bipartite subgraph. MAXCUT is known to be an NP-complete problem and thus one cannot compute \( \text{OPT} \) exactly in polynomial time. On the other hand, it is easy to produce a partition which has at least \( \frac{1}{2} \text{OPT} \) edges across the partition. A breakthrough result of Goemans and Williamson gives a polynomial time approximation that produces a partition which has at least \( 0.878 \text{OPT} \) edges across. Frieze and Kannan proved the following theorem.

Theorem 9. For every \( \delta \) there exists a constant \( c \) such that the following holds for every \( \varepsilon \). For every \( n \)-vertex graph \( G \) having at least \( \delta n^2 \) edges, one can find a partition \( V = V_1 \cup V_2 \) of its vertex set for which \( e(V_1, V_2) \geq \text{OPT} - \varepsilon n^2 \) in time \( 2^{O(1/\varepsilon^2)} \).

Note that the time does not depend on the size of the input.

2.2. Duke, Lefmann, and Rödl. Given a \( k \)-partite graph \( G = (V, E) \) with \( k \)-partition \( V = V_1 \cup V_2 \cup \cdots \cup V_k \), consider a partition \( K \) of \( V_1 \times \cdots \times V_k \) into cylinders \( K = W_1 \times \cdots \times W_k \) where \( W_i \subseteq V_i \) for \( i = 1, 2, \ldots, k \). Define \( V_i(K) = W_i \) for all \( i = 1, 2, \ldots, k \). A cylinder \( K \) is \( \varepsilon \)-regular if all the \( \binom{k}{2} \) pairs of subsets \( (W_i, W_j) \) for \( 1 \leq i < j \leq k \) are \( \varepsilon \)-regular. The partition \( K \) is cylinder-\( \varepsilon \)-regular if all but an \( \varepsilon \)-fraction of the \( k \)-tuples \( (v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k \) are in \( \varepsilon \)-regular cylinders in the partition \( K \).

The weak regularity lemma of Duke, Lefmann, and Rödl is as follows.

Theorem 10. Let \( 0 < \varepsilon < \frac{1}{2} \) and \( \beta = \varepsilon^{k^2 \varepsilon^{-5}} \). Suppose that \( G = (V, E) \) is a \( k \)-partite graph with \( k \)-partition \( V = V_1 \cup \cdots \cup V_k \). Then there exists an cylinder-\( \varepsilon \)-regular partition \( K \) of \( V_1 \times \cdots \times V_k \) into at most \( \beta^{-1} \) parts such that for each \( K \in K \) and \( i \in [k] \), we have \( |V_i(K)| \geq \beta |V_i| \).

Using this form of the regularity lemma, they proved the following result.

Theorem 11. For each graph \( H \) and positive real \( \varepsilon \), if \( n \) is sufficiently large, then for every \( n \)-vertex graph \( G \) one can approximate the number of copies of \( H \) in \( G \) with additive error at most \( \varepsilon \binom{n}{k} \) in time \( O(n^3) \).
The actual result is stronger than what is stated here. See [2] for the original statement. The proof of Theorem 11 is quite straightforward once we have an ‘algorithmic version’ of the Duke-Lefmann-Rödl weak regularity lemma. Here is the proof strategy assuming the algorithmic version. Suppose that \( H \) is a graph on \( k \) vertices and \( G \) be a graph on \( n \) vertices. Suppose that \( G \) contains \( M \) copies of \( H \). Denote the vertex set of \( H \) as \([k]\). Take a random \( k\)-partition \( V_1 \cup \cdots \cup V_k \) of \( G \). Standard estimates show that with high probability the number of copies of \( H \) that have vertex \( i \) in \( V_i \) will be \( \frac{1}{k^h}M \pm \varepsilon n^k \). Hence it suffices to estimate the number of copies of \( H \) where vertex \( i \) embeds to vertex \( V_i \). Now apply the weak regularity lemma to \( V_1 \cup V_2 \cup \cdots \cup V_k \) to obtain a cylinder-\( \varepsilon \)-regular partition \( K \) of \( V_1 \times \cdots \times V_k \). For each \( K \in \mathcal{K} \), we can easily estimate the number of copies of \( H \) having vertex \( i \) in \( V_i(K) \) (it will be the product of ‘relevant’ densities). By summing up all these estimates, one can obtain the desired estimate on \( M \).

### 3. Strong regularity lemma

**Definition 12.** For an equitable partition \( \mathcal{A} = \{V_i \mid 1 \leq i \leq k\} \) of \( V(G) \) and an equitable refinement \( \mathcal{B} = \{V_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq \ell\} \) of \( \mathcal{A} \), we say that \( \mathcal{B} \) is \( \varepsilon \)-close to \( \mathcal{A} \) if the following is satisfied. All pairs \( 1 \leq i \leq i' \leq k \) but at most \( \varepsilon k^2 \) satisfy the following: for all but at most \( \varepsilon \ell^2 \) of pairs \( j, j' \) satisfying \( 1 \leq j, j' \leq \ell \), we have \( |d(V_i, V_{i'}) - d(V_{i,j}, V_{i,j'})| < \varepsilon \).

The strong regularity lemma of Alon, Fischer, Krivelevich, and Szegedy states the following.

**Theorem 13.** For every function \( f : \mathbb{N} \to (0,1) \), there exists an integer \( S = S(f) \) with the following property. For every graph \( G = (V,E) \), there is an equitable partition \( \mathcal{A} \) of the vertex set \( V \) and an equitable refinement \( \mathcal{B} \) of \( \mathcal{A} \) with \( |\mathcal{B}| \leq S \) such that the partition \( \mathcal{A} \) is \( f(1) \)-regular, the partition \( \mathcal{B} \) is \( f(|\mathcal{A}|) \)-regular, and \( \mathcal{B} \) is \( f(1) \)-close to \( \mathcal{A} \).

A crucial feature is that the regularity parameter of \( \mathcal{B} \) depends on the number of parts of \( \mathcal{A} \). This provides a more fine-tuned control.

For ‘reasonable’ functions \( f \), the upper bound on \( S(f) \) is at least wowzer in a power of the inverse of \( \varepsilon = f(1) \). Recall that the tower function is defined by \( T(1) = 2 \) and \( T(n) = 2^{T(n-1)} \). The wowzer-type function is defined by \( W(1) = 2 \) and \( W(n) = T(W(n-1)) \).

The strong regularity lemma was developed to prove the induced graph removal lemma. For a graph \( H \) and a natural number \( k \), we say that a graph \( G \) is \( k \)-far from being induced \( H \)-free if at least \( k \) edges has to be added to or deleted from \( G \) to make it induced \( H \)-free.

**Theorem 14.** *(Induced graph removal lemma)* For any graph \( H \) on \( h \) vertices and \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon, H) > 0 \) such that if a graph \( G \) on \( n \) vertices is \( \varepsilon n^2 \)-far from being induced \( H \)-free, then it contains at least \( \delta n^h \) induced copies of \( H \).
One can try to mimic the proof of the removal lemma based on the regularity lemma, but there are many difficulties that comes up. The main lemma used in overcoming this difficulty was proved using the strong regularity lemma.

**Theorem 15.** For each $0 < \varepsilon < \frac{1}{3}$, there exists $\delta = \delta(\varepsilon)$ such that every sufficient large graph $G = (V,E)$ has an equitable partition $V = V_1 \cup \cdots \cup V_k$ and vertex subsets $W_i \subseteq V_i$ such that $|W_i| \geq \delta|V|$, each pair $(W_i, W_j)$ with $1 \leq i \leq j \leq k$ is $\varepsilon$-regular, and all but at most $\varepsilon k^2$ pairs $1 \leq i \leq j \leq k$ satisfy $|d(V_i, V_j) - d(W_i, W_j)| \leq \varepsilon$.

Note that in the statement above, all pairs $(W_i, W_j)$ are $f(k)$-regular (even for $i = j$). So far we have been only discussing the regularity of distinct pairs. For a set $X \subseteq V(G)$, we say that $(X, X)$ is $\varepsilon$-regular if for every pair of disjoint subsets $A \subseteq X$ and $B \subseteq X$ of sizes $|A| \geq \varepsilon |X|$ and $|B| \geq \varepsilon |Y|$, we have $|d(A, B) - d(X, X)| \leq \varepsilon$. It is not too difficult to prove the induced removal lemma using Theorem 15 (one must develop the induced counting lemma).

**Proof of Theorem 15 (weak version).** We prove a version of Theorem 15 where we do not impose the pairs $(W_i, W_i)$ to be $\varepsilon$-regular.

Define $f(k) = \frac{\varepsilon^2}{k^2}$ for all $k$. Apply Theorem 13 to find an equitable partition $\mathcal{A}$ and an equitable refinement $\mathcal{B}$ with $|\mathcal{B}| \leq S$ such that the partition $\mathcal{A}$ is $\varepsilon$-regular, the partition $\mathcal{B}$ is $f(|\mathcal{A}|)$-regular, and $\mathcal{B}$ is $\varepsilon$-close to $\mathcal{A}$. Let $\mathcal{A} = \{V_i | 1 \leq i \leq k\}$ and $\mathcal{B} = \{V_{ij} | 1 \leq i \leq k, 1 \leq j \leq \ell\}$.

Call a function $\sigma : [k] \to [\ell]$ bad if (i) there exists a pair $(V_{i, \sigma(i)}, V_{i', \sigma(i')})$ that is not $f(k)$-regular or (ii) there exists at least $2\varepsilon k^2$ pairs $(i, i')$ such that $|d(V_{i, \sigma(i)}, V_{i', \sigma(i')})| \geq \varepsilon$. The number of functions that are bad because of (i) is at most

$$\binom{k}{2} \cdot f(k) \varepsilon^2 \cdot \ell^{k-2} < \varepsilon \cdot \ell^k.$$ 

On the other hand, if there are $M$ functions that are bad because of (ii), then

$$M \cdot 2\varepsilon k^2 \leq \varepsilon k^2 \cdot \ell^k + \binom{k}{2} \cdot \varepsilon \ell^2 \cdot \ell^{k-2},$$

and thus $M \leq \frac{3\ell}{4^4}$. Hence there exists at least one function $\sigma$ that is not bad. Define $W_i = V_{i, \sigma(i)}$ for all $i \in [k]$ and note that $W_i$ has the desired properties.

To prove the general case, one can instead define $W_i = V_{i, a_1} \cup V_{i, a_2} \cup \cdots \cup V_{i, a_t}$ for appropriately chosen indices $a_1, a_2, \ldots, a_t$. \qed

**References**

[1] D. Conlon and J. Fox, Bounds for graph regularity and removal lemmas.