LECTURE 1. TURÁN'S THEOREM

1. Turán's theorem

Extremal combinatorics studies the maximum (or minimum) size of a discrete object under certain constraints. For example, a basic question one can ask is

> What is the maximum number of edges in a triangle-free graph?

This basic question was first studied by Mantel in 1907. Throughout this course we will study various questions of a similar flavour and learn how the field developed around these foundational results.

For two graphs H and G, we say that G *contains* H *as a subgraph*, or G *contains a copy of* H, if there exists an injective map $f : V(H) \to V(G)$ such that $\{f(v), f(w)\}$ is an edge of G if $\{v, w\}$ is an edge of H. We say that G *contains* H *as an induced subgraph* if there exists an injective map as above such that $\{f(v), f(w)\}$ is an edge of G if and only if $\{v, w\}$ is an edge of H. Throughout the course we will mostly discuss non-induced subgraphs.

Theorem 1. *(Mantel) If G is an n-vertex triangle-free graph, then it has at most $\frac{n^2}{4}$ edges.*

Proof 1. Let G be a graph with n vertices and m edges. For an edge $\{x, y\}$, note that $d(x) + d(y) \le n$ since every vertex $z \neq x, y$ can be adjacent to at most one vertex out of x and y. Let $d(v)$ be the degree of a vertex v. Note that

$$\sum_{v \in V} d(v)^2 = \sum_{\{x,y\} \in E} (d(x) + d(y)) \le mn.$$

By Cauchy-Schwarz inequality, we have

$$\sum_{v \in V} d(v)^2 \ge \frac{1}{n} \left(\sum_{v \in V} d(v) \right)^2 = \frac{4m^2}{n}.$$

Therefore it follows $\frac{4m^2}{n} \le mn$, which implies $m \le \frac{1}{4} n^2$. □

Proof 2. Let A be a largest independent set in the graph G. Since the neighborhood of every vertex v is an independent set, we must have $d(v) \le |A|$ for all $v \in V$. Let B be the complement of A and note that every edge of G meets B. Therefore

$$m \le \sum_{v \in B} d(v) \le |A||B|,$$

and this implies $m \le \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ by the AM-GM inequality. □

The second proof additionally classifies all n-vertex triangle-free graphs having the maximum number of edges since the only way the inequality can hold is if all vertices in A have exactly $|B|$ neighbors. This implies that G forms a complete bipartite graph.

Definition 2. *A proper vertex-coloring of a graph is a coloring of its vertex set for which each color class forms an independent set.*

The chromatic number of a graph G, which we denote $\chi(G)$, is the minimum number of colors needed in a proper coloring of G. A graph is r-partite if it has chromatic number at most r.

Let $T_{n,r}$ be the complete r-partite graph on n vertices with part sizes being $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$. We will often refer to this graph as the *Turán graph*. Note that the Turán graph is K_{r+1}-free. This is true since for every $(r+1)$-tuple of vertices, by pigeon-hole principle, there must be two that lie in the same part. Hence there is no clique of size $r+1$.

The natural generalization of Mantel's theorem to larger complete graphs was proved by Turán in 1941. Let K_r be the complete graph on r vertices. For a graph G, let $e(G)$ be the number of edges of G.

Theorem 3. *(Turán) If G is an n-vertex K_{r+1}-free graph, then it contains at most $e(T_{n,r})$ edges.*

Proof 1. We prove by induction on n. The theorem is trivially true for $n = 1, 2, \cdots, r$. Assume that it holds for all values less than n. Let G be a K_{r+1}-free graph on n vertices with the maximum number of edges. We may assume that it contains K_r as otherwise we could add edges to G, contradicting maximality.

Let A be a clique of size r and let B be its complement. Each vertex $v \in B$ has at most $r-1$ neighbors in A, and therefore

$$e(G) \leq \binom{r}{2} + (r-1)|B| + e(B).$$

By the inductive hypothesis, $e(B) \leq \prod_{i,j\in[r],i<j} n_i n_j$, where $\sum_{i=1}^{r} n_i = n-r$. One can easily check that

$$\binom{r}{2} + (r-1)|B| + e(B) = \prod_{i,j\in[r],i<j} (n_i+1)(n_j+1),$$

and therefore the statement for n follows. □

Proof 2. (Zykov's symmetrization) Let G be a K_{r+1}-free graph on n vertices with the maximum number of edges. Suppose that xy, yz are non-edges but xz is an edge. If $d(y) < d(x)$, then we can replace y with a 'clone' of x and obtain a K_{r+1}-free graph with more edges. Therefore $d(y) \geq d(x)$. Similarly $d(y) \geq d(z)$. Let $X = V \setminus \{x, y, z\}$ Note that since xz is an edge, we have

$$e(G) \leq e(X) + d(x) + d(y) + d(z) - 1.$$

However, the graph obtained by replacing both x and z with clones of y is K_{r+1}-free and contains at least

$$e(X) + 3d(y) \geq e(X) + d(x) + d(y) + d(z)$$

edges. Therefore it contradicts the maximality of G. This shows that if xy, yz are non-edges, then xz is also a non-edge.

Hence the vertices of G can be partitioned into equivalence classes where vertices in the same class are non-adjacent and vertices in different classes are adjacent. Since G is K_{r+1}-free, it must be a complete s-partite graph for some $s \leq r$. $T_{n,r}$ is the unique graph that maximizes the number of edges among such graphs. □

Proof 3 [**?**]. Consider the 'complement' version. For simplicity, assume that n is divisible by r. We are trying to prove that if a graph has fewer than $\binom{n}{2} - e(T_{n,r}) = r\binom{n/r}{2}$ edges, then it contains an independent set of size at least $r + 1$.

Take a uniform random ordering of the vertices. Let X be the collection of vertices v with no neighbors preceding itself in the ordering. Note that X is an independent set. Since

$$\mathbb{E}[|X|] = \sum_{v \in V} \frac{1}{d(v) + 1} \geq n \cdot \frac{1}{(2m/n) + 1} > r,$$

it follows that there exists an independent set of size at least $r + 1$. □

Turán's theorem is the most fundamental theorem in extremal combinatorics and has plenty of applications. The following simple application is from [**?**].

Theorem 4. *(Erdős) Let S be a set of n points in the plane having diameter at most one. Then the number of pairs of points of distance greater than $\frac{1}{\sqrt{2}}$ is at most $n^2/3$.*

Proof. Consider an auxiliary graph G over the vertex set S where $x, y \in S$ are adjacent if and only if $d(x, y) > \frac{1}{\sqrt{2}}$. Suppose that G contains a K_4 with points x, y, z, w. One can always find three points x, y, z which form an obtuse triangle. However, this contradicts the fact that S has diameter at most one. Therefore G is K_4-free. By Turán's theorem, G contains at most $\frac{n^2}{3}$ edges. □

One can easily check that the bound $\frac{n^2}{3}$ is tight.

2. General graphs

Definition 5. *For a graph H and an integer n, let $ex_H(n)$ be the maximum number of edges in an n-vertex H-free graph, and define $\pi_H(n) = \frac{ex_H(n)}{\binom{n}{2}}$. Define the* Turán number *of H as $\pi_H = \lim_{n\to\infty} \pi_H(n)$.*

We must first check that the limit exists. To see this, let G be an n-vertex H-free graph. Then since each induced subgraph on $n-1$ vertices contains at most $ex(H, n-1)$ edges, it follows that

$$\sum_{X \subset V(G), |X|=n-1} e(X) \leq n \cdot ex(H, n-1).$$

In the left-hand-side, each edge is counted exactly $n-2$ times. Therefore the left-hand-side equals $(n-2)e(G)$ and it follows that

$$(n-2)e(G) \leq n \cdot ex(H, n-1),$$

which is equivalent to

$$\frac{e(G)}{\binom{n}{2}} \leq \frac{ex(H, n-1)}{\binom{n-1}{2}} = \pi_H(n-1).$$

Since the above holds for all n-vertex H-free graphs G, we see that $\pi_H(n) \leq \pi_H(n-1)$, and thus the limit exists.

In 1946, Erdős and Stone discovered the remarkable fact that $\pi_H = 1 - \frac{1}{\chi(H)-1}$.

Theorem 6. *(Erdős-Stone)* $\pi_H = 1 - \frac{1}{\chi(H)-1}$.

The lower bound $\pi_H \geq 1 - \frac{1}{\chi(H)-1}$ easily follows from considering the complete $\chi(H)$-partite graph. Note that we asymptotically recover Turán's theorem when $H = K_{r+1}$. More generally, Erdős and Stone's theorem asymptotically determines $ex(n, H)$ for all non-bipartite graphs H.

When H is a bipartite graph, i.e., when $\chi(H) = 2$, Erdős and Stone's theorem asserts that $\pi_H = 0$. In other words, for each fixed positive real ε, there exists n_0 such that for $n > n_0$, if G is an n-vertex graph with at least εn^2 edges, then G contains H as a subgraph. Thus bipartite graphs are 'degenerate cases' in extremal problems for graphs and turns out to be the most difficult cases. We will come back to this later.

Throughout the course I will present several proofs of the Erdős-Stone theorem. Note that it suffices to prove the theorem for complete r-partite graphs, since every r-partite graph can be embedded into a complete r-partite graph.

The following lemma is the first ingredient in the proof.

Lemma 7. *For all positive reals ε, δ satisfying $\varepsilon < \delta$, there exists n_0 such that the following holds for $n \geq n_0$. Let G be an n-vertex graph with at least $\delta \frac{n^2}{2}$ edges. Then there exists a subgraph G' of minimum degree at least $(\delta - \varepsilon)|V(G')|$ with $|V(G')| \geq \frac{\varepsilon}{2} n$.*

Proof. Let $G_0 = G$. We will iteratively construct graphs G_i for $i = 1, 2, \cdots$ so that we obtain a graph with the desired property in the end.

If G_i has minimum degree at least $(\delta - \varepsilon)|V(G_i)|$, then we are done. Thus assume otherwise. Let G_{i+1} be the graph obtained from G_i by removing an arbitrary vertex with fewer than $(\delta - \varepsilon)(n-i)$ neighbors.

Note that at each time t,

$$e(G) \le e(G_t) + \sum_{i=1}^{t-1} (\delta - \varepsilon)(n-i) \le \frac{(n-t)^2}{2} + (\delta-\varepsilon)\frac{n^2-(n-t)^2}{2} + \frac{t}{2} \le (\delta-\varepsilon)\frac{n^2}{2} + (1-\delta+\varepsilon)\frac{(n-t)^2}{2} + \frac{t}{2}. \tag{1}$$

Since $e(G) \ge \delta\frac{n^2}{2}$, we see that

$$\frac{\varepsilon}{2}n^2 \le (1-\delta+\varepsilon)\,\frac{(n-t)^2}{2} + \frac{t}{2}.$$

Thus if ε is small enough and n is large enough, then $(n-t)^2 > \frac{\varepsilon}{4}n^2$. This implies that $n - t > \frac{\sqrt{\varepsilon}}{2}n$. Hence when the process ends, there still exists at least $n' \ge \frac{\sqrt{\varepsilon}}{2}n$ remaining vertices, on which G induces a subgraph of minimum degree at least $(\delta-\varepsilon)n'$. □

We can now prove the Erdős-Stone's theorem.

Proof of Erdős-Stone theorem. Let G be an n-vertex graph with at least $(1-\frac{1}{r}+2\varepsilon)\frac{n^2}{2}$ edges. By Lemma **??**, there exists a subgraph on $n' \ge \frac{\sqrt{\varepsilon}}{2}n$ vertices of minimum degree at least $(1-\frac{1}{r}+\varepsilon)n'$. Note that n' is sufficiently large as long as n is sufficiently large.

We will prove by induction on s that for each t, if n is large enough, then there exists a complete s-partite graph with t vertices in each part, for $s = 1, 2, \cdots, r+1$. The statement clearly holds for $s = 1$.

For the inductive step, assume that we found a complete s-partite graph with vertex sets $A_1, \cdots, A_s$ for some $s \le r$ where $|A_i| = T$ for $T := \lceil \frac{4t}{r\varepsilon} \rceil$. Define $A = A_1 \cup \cdots \cup A_s$. Count the number of $(s+1)$-tuple of vertices $(v_1, \cdots, v_s, w)$ where $v_i \in A_i \cap N(w)$ for all $i = 1, 2, \cdots, s$. Since $s \le r$, by the minimum degree condition of G, for each fixed s-tuple $(v_1, \cdots, v_s) \in A_1 \times \cdots \times A_s$, there exists at least $r\varepsilon n - |A| \ge \frac{r\varepsilon}{2}n$ choices of w. Hence the total number of $(s+1)$-tuples as above is at least $\frac{r\varepsilon}{2}n\prod_{i=1}^{s}|A_i|$. Let $W \subseteq V \setminus A$ be the set of vertices contained in at least $\frac{r\varepsilon}{4}\prod_{i=1}^{s}|A_i|$ such $(s+1)$-tuples. Note that

$$\frac{r\varepsilon}{2}n\prod_{i=1}^{s}|A_i| \le |W|\cdot\prod_{i=1}^{s}|A_i| + n\cdot\frac{r\varepsilon}{4}\prod_{i=1}^{s}|A_i|,$$

and therefore $|W| \ge \frac{r\varepsilon}{4}n$. Note that each vertex $w \in W$ has at least $\frac{r\varepsilon}{4}|A_i| \ge t$ neighbors in A_i, as otherwise the number of $(s+1)$-tuples as above containing w is less than $\frac{r\varepsilon}{4}\prod_{i=1}^{s}|A_i|$.

Since the number of t-element subsets of A_i is $\binom{T}{t}$, there are $\binom{T}{t}^s$ ways to choose one t-element subset from each set A_i for $i = 1, 2, \cdots, s$. Therefore by the pigeon-hole principle, we can find a subset $B_{s+1} \subseteq W$ of size at least

$\binom{T}{t}^{-s}|W|$ and subsets $B_i \subseteq A_i$ of sizes $|B_i| = t$ for $i = 1, 2, \cdots, s$ such that $B_1, \cdots, B_{s+1}$ forms a complete $(s+1)$-partite graph. □

One can in fact take $t = c \log n$ in the proof above, where c is a constant depending on r and ε. This is known to be the correct order of magnitude.

3. Further study

In this section, we study the question, "What happens around the extremal density?" from various perspectives.

3.1. **Stability.** Informally, a stability-type result says that if a K_{r+1}-free graph has close to the maximum possible number of edges, then it has to be close to being an r-partite graph.

The following stability result for Turán's theorem was proved by Erdős and Simonovits.

Theorem 8. *(Erdős and Simonovits) For every $r \geq 3$ and every $\varepsilon > 0$, there exists δ and n_0 such that every K_{r+1}-free graph on $n \geq n_0$ vertices with $e(G) \geq (1 - \frac{1}{r} - \delta)\frac{n^2}{2}$ contains disjoint subsets of vertices $A_1 \cup A_2 \cup \cdots \cup A_r$ where each A_i is an independent set, and $|A_1 \cup \cdots \cup A_r| \geq (1-\varepsilon)n$.*

Proof. A slight variation of the proof of Lemma **??** (Exercise) shows that there exists a subgraph $G' \subseteq G$ on $n' \geq (1 - \varepsilon')n$ vertices with minimum degree at least $(1 - \frac{1}{r} - \frac{\delta}{2})n'$.

If δ is small enough, then G' contains a copy of K_r; suppose that the set $X = \{x_1, x_2, \cdots, x_r\}$ forms a clique. Note that each vertex $v \notin X$ is adjacent to at most $r-1$ vertices of X. Let $A \subseteq V \setminus X$ be the vertices adjacent to $r-1$ vertices and $B = V \setminus (X \cup A)$. Then

$$e(X, V \setminus X) \leq (r-1)|A| + (r-2)(n' - r - |A|).$$

On the other hand, the minimum degree condition implies that

$$\begin{aligned} e(X, V \setminus X) &\geq r \cdot \left(1 - \frac{1}{r} - \delta\right) n' - r(r-1) \\ &= (r - 1 - r\delta)n' - r(r-1). \end{aligned}$$

Hence

$$|A| + (r-2)(n' - r) \geq (r - 1 - r\delta)n' - r(r-1),$$

and

$$|A| \geq (1 - r\delta)n' + r.$$

For each $i = 1, 2, \cdots, r$, let $A_i \subseteq A$ be the vertices incident to all vertices in $X \setminus \{x_i\}$. Note that $A = A_1 \cup \cdots \cup A_r$. Suppose that A_1 contains an edge yz. Then the vertices $\{y, z, x_2, x_3, \cdots, x_r\}$ forms a clique of size $r+1$ and contradicts our assumption. Hence A_1 is an independent set. Similarly, A_i is an independent set for all $i = 1, 2, \cdots, r$. □

A slightly different form of stability result was established by Andrásfai, Erdős, and Sós.

Theorem 9. *If G is an n-vertex triangle-free graph with minimum degree greater than $\frac{2}{5}n$, then it is bipartite.*

To see where the threshold $\frac{2}{5}n$ comes from, we consider *blow-up* of graphs. A blow-up of a graph H is a graph obtained from H by replacing each vertex $v \in V(H)$ by an independent set U_v, where for two vertices v and w, the pair U_v and U_w forms a complete bipartite graph if $\{v, w\}$ is an edge of H, and forms an empty bipartite graph if $\{v, w\}$ is not an edge of H. Note that all extremal graphs for the Turán problem are blow-up of complete graphs. We say that a blow-up is *balanced* if $|U_v| = |U_w|$ for all $v, w \in V(H)$. Now the tightness of $\frac{2}{5}n$ in the theorem above can be seen by considering the n-vertex balanced blow-up of C_5, a cycle on 5 vertices, since it is triangle-free and has minimum degree exactly $\frac{2}{5}n$.

Proof. Suppose that G is not bipartite. Then it contains an odd cycle. Let $(x_1, x_2, \cdots, x_k)$ be an odd cycle of minimum length. The given condition implies that $k \geq 5$. Let $X = \{x_1, x_2, \cdots, x_k\}$. Note that the subgraph of G induced on X is an odd cycle (no chords), since otherwise one can find an odd cycle of length less than k. Therefore, the number of edges within X is exactly $2k$. If a vertex $v \notin X$ is adjacent to at least three vertices in X, then we can find an odd cycle of length less than k. Hence the number of edges between X and $V \setminus X$ is at most $2|V \setminus X|$.

Hence

$$\sum_{i=1}^{k} d(x_i) \leq 2k + 2(n-k) = 2n.$$

This is a contradiction since the minimum degree condition implies

$$\sum_{i=1}^{k} d(x_i) > \frac{2k}{5}n \geq 2n.$$ □

The result of Andrásfai, Erdős, and Sós is in fact more general and states that every K_r-free graph with minimum degree greater than $\frac{3r-7}{3r-4}n$ has chromatic number at most $r-1$. The proof was later simplified by Brandt [**?**]. Note that the Erdős-Simonovits theorem follows as a corollary of the Andrásfai-Erdős-Sós theorem, given the modified version of Lemma **??** (which was used in the first part of the proof of the Erdős-Simonovits theorem).

There is a fascinating theory of K_r-free graphs of large minimum degree. For triangles, Hajnal proved that for every $\varepsilon > 0$, there are n-vertex triangle-free graphs with arbitrarily large chromatic number and minimum degree at least $(\frac{1}{3} - \varepsilon)n$. On the other hand, Brandt and Thomassé showed that triangle-free graphs of minimum degree greater than $\frac{n}{3}$ have chromatic number at most 4 thus showing a striking dichotomy. Using this result, Goddard and Lyle showed that K_r-free graphs of minimum degree greater than $\frac{2r-5}{2r-3}n$ has chromatic number at most $r+1$. Similarly as in the triangle case, the chromatic number can be arbitrarily large if the minimum degree is at most $(\frac{2r-5}{2r-3} - \varepsilon)n$.

There are many interesting open problems that ask questions about the structure of triangle-free graphs. Erdős offered a \$250 prize for the solution of the following problem.

Conjecture 10. *(Erdős) Any triangle-free graph on n vertices should contain a set of $\frac{n}{2}$ vertices that spans at most $\frac{n^2}{50}$ edges.*

Here is another long-standing open problem of Erdős.

Conjecture 11. *(Erdős) Any triangle-free graph on n vertices can be made bipartite by removing at most $\frac{n^2}{25}$ edges.*

Interestingly in both problems, the balanced blow-up of C_5 is the conjectured extremal graph.

3.2. **Multiplicity.** What happens above the extremal number? That is, if a graph has more than $ex(n, H)$ edges, then how many copies of H must it contain?

Proposition 12. *For every r and ε, there exists δ such that the following holds. If G is an n-vertex graph with at least $\left(1 - \frac{1}{r} + \varepsilon\right) \frac{n^2}{2}$ edges, then it contains at least δn^{r+1} copies of K_{r+1}.*

Proof. By Lemma **??**, there exists a subgraph $G' \subseteq G$ on $n' \geq \frac{\sqrt{\varepsilon}}{4} n$ vertices of minimum degree at least $\left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right) n'$. In G', note that each r-tuple of vertices has at least $\frac{\varepsilon r}{2} n'$ common neighbors. Therefore if we greedily construct a copy of K_{r+1}, then at each step we have at least $\frac{\varepsilon r}{2} n'$ choices. This shows that the number of copies of K_{r+1} in G' is at least

$$\left(\frac{\varepsilon r}{2} n'\right)^{r+1} \geq \frac{\varepsilon^{2r+2} r^{r+1}}{8^{r+1}} n^{r+1}. \qquad \square$$

The main message of Proposition **??** is :

> If an n-vertex graph has significantly more edges than the extremal number $ex(n, H)$, then constant fraction of $|V(H)|$-tuple of vertices forms a copy of H.

Such a phenomenon is known as *supersaturation* and holds for many extremal problems. Proving the fact for general H will be given as an exercise.

The dependency between δ and ε given by Proposition **??** is far from being optimal and establishing the correct order of dependency was a famous open problem of Lovász and Simonovits that drawn a lot of attention. Their conjecture stated that the number of copies of K_{r+1} in an n-vertex graph G with $(1 - \frac{1}{r} + \varepsilon)\binom{n}{2}$ edges is minimized when G is a certain complete $(r+1)$-partite graph. After a series of work by Goodman, Lovász, Simonovits, Bollobás, Fisher, Razborov, and Nikiforov, recently Reiher solved the conjecture.

4. Open problems

4.1. **Turán density for hypergraphs.** A k-uniform hypergraph H is a pair (V, E) of finite sets where $E \subseteq \binom{V}{k}$, where $\binom{V}{k}$ is the set of all subsets of V of size k. We refer to the elements in E as *hyperedges*, or *edges*, of H. A t-vertex complete k-uniform hypergraph is $K_t^{(k)}$ the hypergraph with vertex set $[t]$ where k-tuples form an hyperedge. Extremal numbers and Turán densities can be defined for hypergraphs just as we did for graphs.

Generalizing Turán's theorem to hypergraphs is one of the most famous open problems in extremal graph theory. There is no single pair (t, k) with $t > k \geq 3$ where the Turán density of $K_t^{(k)}$ is known. In fact, Erdős offered \$1000 for computing the Turán density all cases, and \$500 for computing any special case. The problem is still open even for $K_4^{(3)}$, the complete 3-uniform hypergraph on 4 vertices.

Conjecture 13. *(Erdős) The Turán density of $K_4^{(3)}$ is $\frac{5}{9}$.*

Unlike in the graph case, there are several different examples achieving the conjectured bound. The current best known bound is given by Razborov, who using the powerful framework of Flag Algebra showed that the Turán density is at most 0.561666. This bound has been established using a computer program and has not been formally written down. The best known published bound is $\frac{3+\sqrt{17}}{12} \approx 0.593593$ by Chung and Lu.

4.2. **Caccetta-Häggkvist conjecture.** The following is a special case of a well-known conjecture of Caccetta and Häggkvist.

Conjecture 14. *Every simple directed n-vertex graph with minimum out degree at least $\frac{n}{3}$ has a directed cycle of length* 3.

Although the given condition is slightly different from Turán's theorem and its extensions, it is of similar flavor. In fact, it is impossible to force a directed cycle of length 3 using only a condition on the number of edges, since there exists an n-vertex simple directed graph with $\binom{n}{2}$ edges that contains no directed cycle.

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