Abstract. This problem set corresponds to the first block of the course, covering material on the basics of Lie groups and Lie algebras. This Problem Set was uploaded Tuesday Oct 10: to this day we have covered up to the equivalence between the category of simply-connected Lie groups and the category of Lie algebras.

Student Name:

Grade: / 100

Instructions: It is perfectly fine to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page. The notation 2.1-2.4 means exercises from 2.1 to 2.4, including 2.2 and 2.3. Start with the problems from the book.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning and precise mathematical expressions and formulas. In this first assignment, please do invest time in writing carefully.


Exercises from the book (90 pts)

Exercise 1. (10 pts) Exercises 2.4-2.5 Pages 21-22.
Exercise 2. (25 pts) Exercises 2.7-2.10 Page 22.
Exercise 5. (25 pts) Exercise 3.5-3.8 Pages 48-49.

The grade distribution reflects the fact that the above exercises are at the core of the course and you should invest most of the time working on them. The subsequent problems contain beautiful ideas but only work on them if you are interested, I would also try them in this order.

Problem Set (10 pts)

Problem 1. Consider the two Lie groups

\[ G_1 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad G_2 = \left\{ \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} : x \in \mathbb{R} \right\} \]

(a) Show that \( G_1 \) and \( G_2 \) are isomorphic to the Lie group \( (\mathbb{R}, +) \).

(b) Draw the orbits of the action of \( G_1 \) onto \( \mathbb{R}^2 \) by matrix multiplication.

(c) Draw the orbits of the action of \( G_2 \) onto \( \mathbb{R}^2 \). What are the stabilizers? The group \( G_2 \) is called the group of hyperbolic rotations. Explain why.
Problem 2. Consider a set of matrices \( \{A_1, \ldots, A_k\} \subseteq \text{End}(V) \).

(a) Suppose that the matrices \( \{A_1, \ldots, A_k\} \) can be simultaneously diagonalized. Show that \([A_i, A_j] = 0\) for all \( i, j = 0, \ldots, k \).

(b) Conversely, assume that \( A_1 \) has distinct eigenvalues and \([A_1, A_2] = 0\). Prove that \( A_1 \) and \( A_2 \) can be simultaneously diagonalized.

(c) Is it true that \([A, B] = 0\) implies that \( A \) and \( B \) can be simultaneously diagonalized?

Problem 3.

(a) Show that the matrix \(
\begin{pmatrix}
-2 & 0 \\
0 & -0.5
\end{pmatrix}
\) \( \in \text{SL}(2, \mathbb{R}) \) is not in a one-parameter subgroup of \( \text{SL}(2, \mathbb{R}) \) but it is on a one-parameter subgroup of \( \text{SL}(2, \mathbb{C}) \).

(b) Show that the matrix \(
\begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix}
\) \( \in \text{SL}(2, \mathbb{C}) \) is not in a one-parameter subgroup of \( \text{SL}(2, \mathbb{C}) \) but it is on a one-parameter subgroup of \( \text{GL}(2, \mathbb{C}) \).

(c) Show that any matrix in \( \text{GL}(n, \mathbb{C}) \) is in a one-parameter subgroup.

Problem 4. Consider the real Lie group \( U(n) \), its Lie algebra \( u(n) \) and the Lie subalgebra \( t = \left\{ \begin{pmatrix} ia_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & ia_n \end{pmatrix} : a_1, \ldots, a_n \in \mathbb{R} \right\} \subseteq u(n) \).

(a) Find a group \( T \subseteq U(n) \) such that \( \text{Lie}(T) = t \).

(b) Can such a group \( T \) be normal in \( U(n) \) ?

(c) Consider the subgroup \( T = \left\{ \begin{pmatrix} ia_1 & 0 & 0 \\
0 & ia_2 & 0 \\
0 & 0 & ia_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0, a_1, a_2, a_3 \in \mathbb{R} \right\} \subseteq \text{SU}(3) \).

Compute its Lie algebra \( t \subseteq \text{su}(3) \).

(d) Show that \( \ker(\exp) \subseteq t \) is a lattice.

(e) Draw this lattice in \( t \cong \mathbb{R}^2 \) using the inner product \( \langle x, y \rangle = -\text{tr}(xy) \).

Problem 5. Consider the following inclusion \( Z = \left\{ \begin{pmatrix} 1 & 0 & n \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \subseteq N = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\
0 & 1 & \gamma \\
0 & 0 & 1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\} \).

(a) Show that \( Z \subseteq N \) is a normal Lie subgroup.

(b) Find the Lie algebras \( \mathfrak{z} \subseteq \mathfrak{n} \) and verify that \( \mathfrak{z} \) is an ideal.

(c) Consider the Hilbert space \( L^2(\mathbb{R}) \) of \( L^2 \)-integrable functions with the transformations \( T_\alpha, M_\beta, C_\gamma : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \),

\[
T_\alpha(f)(x) = f(x - \alpha), \quad M_\beta(f)(x) = e^{2\pi i \beta x} f(x), \quad C_\gamma(f) = e^{2\pi i \gamma} f(x).
\]

Show that the Lie group \( N/Z \) is isomorphic to transformations of the form \( T_\alpha M_\beta C_\gamma \).
Problem 6. (Optional) Let $M$ be a smooth manifold.

(a) Show that the map $\exp : \text{Vect}(M) \rightarrow \text{Diff}(M)$ assigning to each vector field $X \in \text{Vect}(M)$ the time-one map $\varphi^1$ of its flow $\varphi^t$ cannot be defined. (Hint: Try $M = \mathbb{R}$.)

(b) Show that $\exp : \text{Vect}(S^1) \rightarrow \text{Diff}(S^1)$ exists.

(c) Show that $\exp : \text{Vect}(S^1) \rightarrow \text{Diff}(S^1)$ is not locally surjective.