1. Holomorphic Bundles on $\mathbb{P}^1$. Let $n \in \mathbb{Z}$ be an integer.

(a) Show that $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is isomorphic to the space of polynomials $p(z)$ of deg $p \leq n$.

(b) Prove that the higher cohomology groups $H^i(\mathbb{P}^1, \mathcal{O}(n))$ vanish, for $i \geq 2$.

(c) Give a necessary and sufficient condition on $n$ for $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$.

(d) Consider the complex surface $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O})$ obtained by fiberwise projectivizing the rank 2 holomorphic vector bundle $\mathcal{O}(n) \oplus \mathcal{O}$ over $\mathbb{P}^1$. Compute the normal bundle of the image of the section $s_\infty: \mathbb{P}^1 \to \mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O})$, $p \mapsto (p, [0 : 1])$.

(e) Find two embeddings $i_1, i_2: \mathbb{P}^1 \to \mathbb{P}^2$ such that the normal bundles $\nu(i_1)$ and $\nu(i_2)$ are respectively $\mathcal{O}(1)$ and $\mathcal{O}(2)$. Does there exists an embedding $i_n: \mathbb{P}^1 \to \mathbb{P}^2$ whose normal bundle is $\mathcal{O}(n)$ for all $n \in \mathbb{N}$?

(f) Give an example of two holomorphic vector bundles $E_1, E_2$ on $\mathbb{P}^1$ which are smoothly trivial but $E_1 \not\cong E_2$ as holomorphic vector bundles.

(g) Compute the normal bundle $\nu(j)$ of the embedding $j: \mathbb{P}^1 \to \mathbb{P}^3$, $[z : w] \mapsto [z^3 : z^2 w : zw^2 : w^3]$, and in particular show that $h^0(\mathbb{P}^1, \nu(j)) = 12$.

2. Adjunction Formula.

(a) Consider a plane algebraic Riemann surface $C \subseteq \mathbb{P}^2$. Show that $K_C \cong (K_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(\deg(C)))|_{C}$.

(b) Deduce from Part (a) that $g(C) = (d - 1)(d - 2)/2$, where $d = \deg(C)$.

(c) Prove that if $\deg(C) = 4$ then $C$ is the image of a canonical embedding$^1$.

(d) Suppose that $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the zero of a bihomogeneous polynomial $p(z_1, w_1, z_2, w_2)$ of bidegree $(a, b)$, compute the genus of $C$.

3. Automorphisms of plane curves.

(a) Show that $\text{Aut}(\mathbb{P}^2) \cong PGL(\mathbb{C}^3)$.

(b) Let $C$ be a plane algebraic Riemann surface $C \subseteq \mathbb{P}^2$. Prove that any $\phi \in \text{Aut}(C)$ is the restriction of a global automorphism $PGL(3, \mathbb{C})$.

$^1$Recall that a canonical embedding is one obtained as $C \to \mathbb{P}(H^0(C, K_C))$.  

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4. **Short exact sequences.**

   a. Show that a short exact sequence of smooth vector bundles splits, i.e.
   \[
   0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0
   \]
   implies the direct sum decomposition \( E_2 \cong E_1 \oplus E_3 \), as smooth vector bundles.

   b. Exhibit a short exact sequence of holomorphic vector bundles which does not split.

   c. Show that \( \deg(E_1 \times E_2) = \text{rk}E_1 \cdot \deg E_2 + \text{rk}E_2 \cdot \deg E_1 \).

5. **Lines in a quintic hypersurface** \( F_5 \subseteq \mathbb{P}^4 \). Consider the hypersurface

   \[
   F_5 = \{ [z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4 : z_2 \cdot p_2 + z_3 \cdot p_3 + z_4 \cdot p_4 = 0 \},
   \]

   where \( p_2, p_3, p_4 \in \mathbb{C}[z_0, z_1, z_2, z_3, z_4] \) are degree 4 homogeneous polynomials.

   a. Consider the line \( C \) given by the embedding
   \[
   i : \mathbb{P}^1 \rightarrow F_5, \quad i([z : w]) = [z : w : 0 : 0 : 0].
   \]
   Show that \( TP^4|_C \) is isomorphic to \( \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \).

   b. Prove that the restriction \( TF_5|_C \) is \( \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus (-3) \).

   c. Conclude that the normal bundle \( \nu(i) \) is either \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \), \( \mathcal{O} \oplus \mathcal{O}(-2) \) or \( \mathcal{O} \oplus \mathcal{O}(-3) \).

   d. (Optional) Choose \( p_2, p_3, p_4 \) where you can deduce that \( \nu(i) \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \).