Abstract. This problem set corresponds to the 6th week of the course, covering Riemann–Roch.

Riemann surfaces are considered to be compact, but are often given by their affine models.

1. a. In class we introduced the two spaces:
   \[ H^0(D) := \{ f : X \to \mathbb{P}^1 \text{ meromorphic : } (f) + D \geq 0 \}, \]
   \[ H^0(K - D) := \{ \eta \in \mathcal{M}_X^1 : (\eta) - D \geq 0 \}. \]
   Show that this notation is correct, that is, the vector space \( H^0(K - D) \) of meromorphic 1–forms \( \eta \) with the constraint \( (\eta) - D \geq 0 \) is isomorphic to the vector space \( H^0(K - D) \) of meromorphic functions \( f : X \to \mathbb{P}^1 \) with \( (f) + K - D \geq 0 \).

b. Conclude that \( H^0(K - D) = 0 \) if the degree of \( D \) is larger than \( 2g + 2 \).

2. Consider a genus–g Riemann surface \( X \), a number \( n \geq 1 \) and a point \( p \in X \).
   a. Show that \( h^0((n + 1)p) \) is either \( h^0(np) \) or \( h^0(np) + 1 \).
   
   b. Show that it \( h^0(np) = n - g + 1 \) if \( n \geq 2g - 1 \).

   c. By the two statements above, there exists an increasing sequence of \( g \) numbers
      \[ 1 = n_1(p) < n_2(p) < \ldots < n_g(p) \leq 2g - 1, \]
      such that \( h^0(n_i(P)p) = h^0((n_i(P) - 1)p) \) for \( i = 1, \ldots, g - 1 \).

   Give an example of a Riemann surface \( X \) with two points \( p_1, p_2 \in X \) whose associated sequences are different.

   d. Show that the number of points whose sequence is not \( (1, 2, \ldots, g) \) is at least \( 2g + 2 \).
      Give an example of a Riemann surface with exactly \( 2g + 2 \) such points, with \( g \geq 2 \).

   e. Show that the number of points whose sequence is not \( (1, 2, \ldots, g) \) is at most \( g^3 - g \).

3. Consider a Riemann surface \( X \) of genus \( g \geq 2 \) and \( \text{Aut}(X) \) its biholomorphism group.
   a. Show that any \( \varphi \in \text{Aut}(X) \) with more than \( 2g + 2 \) fixed points is the identity.

   **Hint**: The number \( 2g + 2 \) appears in Problem 1 as a lower bound for the number of points with a particular property, you might want to study the action of \( \varphi \) on these points.

   b. Conclude that \( |\text{Aut}(X)| \) is finite.

   Note that using the PSet from last week we then get the bound \( |\text{Aut}(X)| \leq 84(g - 1) \).
4. Consider the Riemann surfaces
\[ C_g(z_1, \ldots, z_{2g+2}) = \{(z, w) \in \mathbb{C}^2 : w^2 = (z - z_1)(z - z_2) \cdots (z - z_{2g+2}) \}. \]
a. Find a basis of the vector space \( H^{1,0}_{C_g} \).

b. Consider the case \( C_1(-1, 0, 1, \infty) \), find a basis of the vector spaces
\[ H^0(P), H^0(2P), H^0(3P), H^0(4P), \]
where \( P = (0, 0) \) is the origin.

5. a. Show that Riemann surface \( X \) with \( g(X) = 1 \) is of the form \( C_1(z_1, z_2, z_3, z_4) \).

b. Show that Riemann surface \( X \) with \( g(X) = 2 \) is of the form \( C_2(z_1, z_2, z_3, z_4, z_5, z_6) \).

c. Show that a genus 3 Riemann surface \( X \) always has a map
\[ f : X \rightarrow \mathbb{P}^1 \]
with either \( \deg(f) = 2 \) or \( \deg(f) = 3 \).

d. Show that Riemann surface \( X \) with \( g(X) \geq 3 \) cannot have two maps
\[ f_2 : X \rightarrow \mathbb{P}^1, \quad f_3 : X \rightarrow \mathbb{P}^1 \]
with degrees \( \deg(f_2) = 2 \) and \( \deg(f_3) = 3 \).

e. Provide an heuristic argument for the existence of Riemann surfaces \( X \) of genus \( g(X) \geq 3 \) with no degree 2 map \( f : X \rightarrow \mathbb{P}^1 \).