Abstract. This problem set corresponds to the second week of the course, covering introductory material on Riemann surfaces. Choose three of the following five problems, the full grade is computed over these three problems. Every additional problem that you submit will increase the grade.

The first problem on isothermal coordinates expands on the lecture about Gauss’ theorem and serves to practice basic differential geometry, but from the perspective of this 18.116 course it is more important to master Problems 2, 3, 4 and 5.

Problem 2 is essentially an advanced problem in a complex analysis class, it reviews and expands the discussions on the group of biholomorphisms we had on Wednesday Sep 14.

Problem 3 is the first (of many) in which we study algebraic curves as Riemann surfaces, as explained on Monday Sep 19, and uses Problem 1 from Problem Set 1.

Problems 4 and 5 serve to practice the perspective of quotient Riemann surfaces as explained on Wednesday Sep 21.

1. Isothermal Coordinates:

In class we proved that a real analytic Riemannian surface \((\Sigma, g)\) admits a holomorphic structure, i.e. it is a Riemann surface. The coordinates \((u(x, y), v(x, y)) \in U \subseteq \mathbb{R}^2\) in which the Riemannian metric

\[
g(x, y) = a(x, y)dx \otimes dx + 2b(x, y)dx \otimes dy + c(x, y)dy \otimes dy
\]

becomes conformally flat

\[
g(u, v) = e^{f(u,v)}(du \otimes du + dv \otimes dv)
\]

are called isothermal. In this exercise we will learn about explicit examples of isothermal coordinates in the sphere, ellipsoids, the flat torus and surfaces of revolution.

a. Let \((S^2, g_{st}|_{S^2}) \subseteq (\mathbb{R}^3, g_{st})\) be the round 2–sphere:

\[
S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}
\]

\[
g_{st}(x, y, z) = dx \otimes dx + dy \otimes dy + dz \otimes dz.
\]

**Show** that at a point \(p = (x_0, y_0, z_0) \in S^2\) there exists a neighborhood \(U_p \subseteq S^2\) such that the projection \(\pi : U_p \rightarrow T_p S^2\) onto the tangent plane \(T_p S^2\) of the antipodal point \(-p = (-x_0, -y_0, -z_0) \in S^2\) provides isothermal coordinates \((u, v) \in \mathbb{R}^2\) around \(p\).

b. Part (a) solves the case of the 2–sphere \(S^2 \subseteq \mathbb{R}^3\), but let us study it further. First, **show** that with the parametrization \(\phi : [0, 2\pi) \times [0, \pi] \rightarrow S^2 \subseteq \mathbb{R}^3\) of the sphere given by spherical coordinates

\[
(\theta, \sigma) \rightarrow (\cos \theta \cos \sigma, \sin \theta \cos \sigma, \sin \sigma), \quad (\theta, \sigma) \in \phi : [0, 2\pi) \times [0, \pi]
\]

The heat equation in the plane is \((\partial_t - \Delta)u(x, y, t) = 0\), and the Laplacian vanishes when applied to the isothermal coordinates, i.e. they are harmonic. Hence they are a steady solution to the heat equation, where the temperature remains constant along time, thus isothermal (same–temperature).
the coordinates \((\theta, \sigma)\) are not isothermal.

Second, let us try to fix this. The factorization
\[
d\theta \otimes d\theta + \frac{d\sigma \otimes d\sigma}{\cos^2 \sigma} = \left(d\theta + i \frac{d\sigma}{\cos \sigma}\right)\left(d\theta - i \frac{d\sigma}{\cos \sigma}\right)
\]
leads to the coordinate change \(\varphi = \ln(\tan \sigma + \sec \sigma)\) such that \(\cos \sigma d\varphi = d\sigma\). Since \(\cos \sigma = \text{sech} \varphi\) in the coordinates \((\theta, \varphi)\) the metric becomes conformally flat.

**Show** that this is indeed correct, i.e. the parametrization
\[
(\theta, \varphi) \mapsto (\cos \theta \text{sech} \varphi, \sin \theta \text{sech} \varphi, \tanh \varphi)
\]
makes \((\theta, \varphi)\) isothermal coordinates, and **find** the conformal factor.

c. The discussion in Part (b) might generalize to ellipsoids. The Mercator coordinates \((\theta, \varphi)\) for an ellipsoid \(E(a, b, c)\) with semiaxis \(a, b, c \in \mathbb{R}^+\), are given by the formula
\[
\phi(\theta, \varphi) = (a \cos \theta \text{sech} \varphi, b \sin \theta \text{sech} \varphi, c \tanh \varphi),
\]
which generalizes Part (b). This parametrization was discovered by the Flemish G. Mercator (1569), **show** however that \((\theta, \varphi)\) are not isothermal coordinates unless \(a = b = c\). Thus Part (b) does not generalize to ellipsoids.

But since you are asking, there are explicit isothermal coordinates
\[
(\xi, \eta, \zeta) \in (-c^2, \infty) \times (-b^2, -c^2) \times (-a^2, -b^2)
\]
for the ellipsoid, called confocal ellipsoidal coordinates, given by
\[
x^2 = \frac{(a^2 + \xi)(a^2 + \eta)(a^2 + \zeta)}{(b^2 - a^2)(c^2 - a^2)},
\]
\[
y^2 = \frac{(b^2 + \xi)(b^2 + \eta)(b^2 + \zeta)}{(a^2 - b^2)(c^2 - b^2)},
\]
\[
z^2 = \frac{(c^2 + \xi)(c^2 + \eta)(c^2 + \zeta)}{(a^2 - c^2)(b^2 - c^2)}.
\]
Note that any triple \((x, y, z)\) determines \((\xi, \eta, \zeta)\) uniquely, but each \((\xi, \eta, \zeta)\) gives \(8 = 2^3\) different points, one in each octant. **Show** that the confocal ellipsoidal coordinates are isothermal coordinates.

d. Let us move on to the smooth torus \(\Sigma_1\).

**Show** that the embedding \(T^2(\theta, \sigma) \rightarrow (\mathbb{R}^4, g_{st})\) given by
\[
\iota : (\theta, \sigma) \mapsto (\cos \theta, \sin \theta, \cos \sigma, \sin \sigma), \quad (\theta, \sigma) \in [0, 2\pi) \times [0, 2\pi),
\]
endows the torus \(T^2\) with the flat metric \(g_{st}(\theta, \sigma) = d\theta \otimes d\theta + d\sigma \otimes d\sigma\).

Note that this is even better than conformally flat. There is a holomorphic structure associated to \(\iota(T^2)\) which is given by \(\mathbb{C}\) quotiented by a lattice \(\Lambda\). **Decide** which lattice \(\Lambda \subseteq \mathbb{C}\) should that be.

e. Let \((\Sigma, g) \subseteq (\mathbb{R}^3, g_{st})\) be a smooth surface of revolution with its induced metric \(g = g_{st}|_{\Sigma}\).

**Show** that there are coordinates \((t, \theta) \in D^2\) such that the metric can be written as
\[
g(t, \theta) = e^{f_1(t)} dt \otimes dt + e^{f_2(t)} d\theta \otimes d\theta.
\]

**Conclude** that there are isothermal coordinates \((s, \theta)\) such that
\[
g(s, \theta) = e^{g(s)}(ds \otimes ds + d\theta \otimes d\theta).\]
2. Biholomorphism groups:

In this exercise we study the biholomorphic symmetries for some of the simplest Riemann surfaces, consisting of open sets $U \subseteq \mathbb{C}$.

a. Let us start with the simplest case $U = \mathbb{C}$.

**Show** that the biholomorphisms $f : \mathbb{C} \to \mathbb{C}$ are linear polynomials, i.e.

$$\text{Aut}(\mathbb{C}) = \{ f : \mathbb{C} \to \mathbb{C} : f(z) = az + b, \ (a, b) \in \mathbb{C}^* \times \mathbb{C} \},$$

the operation in the group is composition.

**Show** that $\text{Aut}(\mathbb{C})$ is isomorphic to the group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : (a,b) \in \mathbb{C}^* \times \mathbb{C} \right\},$$

where operation is matrix multiplication.

b. Let us remove the origin from $\mathbb{C}$ and consider $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

**Show** that the biholomorphisms $f : \mathbb{C}^* \to \mathbb{C}^*$ are

$$\text{Aut}(\mathbb{C}) = \{ f : \mathbb{C} \to \mathbb{C} : f(z) = az \text{ or } f(z) = a/z, \ a \in \mathbb{C}^* \},$$

i.e. rotations, scaling and the inversion.

**Compute** the biholomorphism group of $\mathbb{C} \setminus \{z_0\}$ for any $z_0 \in \mathbb{C}$.

c. **Show** that the biholomorphism group $\text{Aut}(\mathbb{C} \cup \{\infty\})$ is isomorphic to

$$\text{PSL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}/\{\text{Id}\},$$

d. The previous biholomorphic groups contained infinitely many elements.

**Compute** the following two automorphism groups

$$\text{Aut}(\mathbb{C} \setminus \{0, 1\}), \ \text{Aut}(\mathbb{C} \setminus \{0, 1, 2.94 - 3.1 \cdot i\}).$$

At this point we can remove more points, **show** however that for generic choice this leads to the trivial group, for instance

$$\text{Aut}(\mathbb{C} \setminus \{0, 1, 2.94 - 3.7i, 5.32 + 9i\}) = \{\text{Id}\}.$$

e. Let us now move on towards the 2–disk $\mathbb{D}^2$. Consider the group of linear transformations

$$\mathbb{C}^2 \to \mathbb{C}^2$$

which preserve the indefinite symmetric form given by

$$\langle (z_1, z_2), (w_1, w_2) \rangle = z_1\bar{w}_1 - z_2\bar{w}_2,$$

and have determinant equal to 1. **Show** that this group is

$$\text{SU}_{1,1}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1, \ a, b \in \mathbb{C} \right\}.$$  

The cool point of this indefinite product is that the unit disk $\mathbb{D}^2 = \{ z \in \mathbb{C} : |z| < 1 \}$ can be expressed as

$$\mathbb{D}^2 \cong \{(z, 1) \in \mathbb{C}^2 : ||(z,1)|| < 0 \} = \left\{ z \in \mathbb{C} : (z,1)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (z,1) < 0 \right\}$$

and in consequence $\text{SU}_{1,1}(\mathbb{C})$ can be seen as acting on $\mathbb{D}^2$ via

$$(z, 1) \mapsto \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az + b \bar{z} + \bar{a} \\ bz + a \end{pmatrix}.$$  

This is just matrix multiplication, which gives $(az + b\bar{z} + \bar{a})$ and then dividing in order to set the second coordinate to 1. Note that both matrices $\pm \text{Id}$ act trivially, and in consequence we have

$$\text{PSU}_{1,1}(\mathbb{C}) = \text{SU}_{1,1}(\mathbb{C})/\{\pm \text{Id}\} \subseteq \text{Aut}(\mathbb{D}^2).$$
Show that the action of $PSU_{1,1}(\mathbb{C})$ on $\mathbb{D}^2$ is transitive and use the Schwarz Lemma, which characterizes biholomorphisms of the disk fixing the origin, to conclude that actually

$$PSU_{1,1}(\mathbb{C}) \cong \text{Aut}(\mathbb{D}^2).$$

Great, we have the biholomorphism group of the unit disk $\mathbb{D}^2$.

f. Consider the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$.

Compute its biholomorphism group $\text{Aut}(\mathbb{H})$ by using Part (d).

Now let us forget we actually knew about $\text{Aut}(\mathbb{D}^2)$ and start with $\mathbb{H}$ from scratch.

Suppose that $f \in \text{Aut}(\mathbb{H})$ extends to an automorphism of its closure $f \in \text{Aut}(\overline{\mathbb{H}})$, sending the boundary $\partial \mathbb{H} = \mathbb{R}$ to itself.

Show that an element $f \in \text{Aut}(\mathbb{H})$ can be extended to an element $\tilde{f} \in \text{Aut}(\mathbb{C} \cup \{\infty\})$ which preserves the real equator $\mathbb{R} \cup \{\infty\} \subseteq \mathbb{C} \cup \{\infty\}$.

This gives a group inclusion $\text{Aut}(\mathbb{H}) \subseteq \text{Aut}(\mathbb{C} \cup \{\infty\})$, and in Part (c) we computed the automorphism group $\text{Aut}(\mathbb{C} \cup \{\infty\}) = PSL(2, \mathbb{C})$.

Conclude that the group of transformations of $\mathbb{H}$ is

$$PSL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \quad a, b, c, d \in \mathbb{R} \right\}.$$

Given that we have a biholomorphism $\mathbb{H} \cong \mathbb{D}^2$, it must be that their biholomorphic symmetries are the same, i.e. there exists a group isomorphism $PSL(2, \mathbb{R}) \cong PSU_{1,1}(\mathbb{C})$.

Find an explicit isomorphism between these two groups.

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Show that the intersection of the two quadrics
\[ Q_1 = \{ [x : y : z : w] \in \mathbb{CP}^3 : x^2 - wz = 0 \}, \]
\[ Q_2^{(a,b)} = \{ [x : y : z : w] \in \mathbb{CP}^3 : y^2 - axz - bz^2 - wx = 0 \} \]
is a smooth Riemann surface if and only if \( 4a^3 + 27b^2 \neq 0 \).

Find the singular points in case the case \( 4a^3 + 27b^2 = 0 \).

b. Find a holomorphic inclusion of the Riemann surface
\[ E^{(a,b)} = \{(u, v) \in \mathbb{C}^2 : v^2 = u^3 + au + b \} \subseteq \mathbb{C}^2 \]
into the intersection \( Q_1 \cap Q_2^{(a,b)} \).

c. It is a healthy practice with polynomials to change viewpoint between coefficients, as in Parts (a) and (b), and roots. Instead of considering the Riemann surfaces \( E^{(a,b)} \) parametrized by \( (a, b) \in \mathbb{C}^2 \), let us consider the family
\[ E^\lambda = \{(u, v) \in \mathbb{C}^2 : v^2 = u(u + 1)(u + \lambda) \} \subseteq \mathbb{C}^2. \]
Prove that we can compactify \( E^\lambda \) to a compact Riemann surface \( \overline{E^\lambda} \) in \( \mathbb{CP}^2 \) by adding a point \( \{ \infty \} \) if \( \lambda \neq 0, 1 \). Show that the Riemann surfaces \( \overline{E^\lambda} \) are intersections of the following two quadrics in \( \mathbb{CP}^3 \):
\[ \{ [x : y : z : w] \in \mathbb{CP}^3 : x^2 + y^2 - wz = 0 \}, \]
\[ \{ [x : y : z : w] \in \mathbb{CP}^3 : (1 - \lambda)x^2 + z^2 - w^2 = 0 \}. \]
d. Let us consider the two quadrics in Part (c), endow their intersection \( \overline{E^\lambda} \) with a commutative group structure. I suggest starting with the affine chart
\[ U_w = \{ [x : y : z : w] \in \mathbb{CP}^3 : w = 1 \} \subseteq \mathbb{CP}^3, \]
which is one of the four \( \mathbb{C}^3 \) that cover \( \mathbb{CP}^3 \). Then study first the affine curve
\[ \overline{E^\lambda} \cap U_w = \{ (x, y, z) : x^2 + y^2 = 1, (1 - \lambda)x^2 + z^2 = 1 \} \]
**Hint:** Start with endowing the circle \( \{ z = \rho e^{i\theta} \in \mathbb{C} : \rho = 1 \} \) with a group structure using a parametrization. Then play the same game with an appropriate parametrization of the curve \( \overline{E^\lambda} \cap U_w \), and finally complete to the compactification \( \overline{E^\lambda} \).

e. Thus far we have dealt with algebraic curves defined by cubic polynomials of a specific form; however, up to a coordinate transformation these are all cubic polynomials.

Show that any polynomial \( p(x, y) \) of degree 3 can be brought to be of the form
\[ v^2 = u(u + 1)(u + \lambda), \] for some \( \lambda \in \mathbb{C} \).

f. It is outstanding for a geometric object to have an algebraic operation, even if it is only a group structure. Thanks to Parts (d) and (e) we have proven that any Riemann surface defined by a cubic equation admits a group structure!

This has many consequences, starting already in the smooth topology of the underlying closed surface: prove that a orientable smooth closed surface in which we have a commutative group structure admits a non–zero vector field, and thus must be diffeomorphic to a torus \( \Sigma_1 \).

**Comment:** In the near future we will be able to compute the genus of Riemann surfaces described as algebraic curves from the degree of their defining polynomials. This will tell us that the only time we obtain (smooth closed) tori in \( \mathbb{CP}^2 \) is for cubic polynomials.
4. Quotients:

In this exercise we study the Riemann surfaces $\mathbb{C}^*$, $\mathbb{D}^*$ and $A_{1,R}$ from the quotient viewpoint.

a. **Show** that the action of $\mathbb{Z}$ on $\mathbb{C}$ given by translation

$$\mathbb{Z} \times \mathbb{C} \rightarrow \mathbb{C}, \quad (n,z) \mapsto z + n,$$

has quotient Riemann surface $\mathbb{C}/\mathbb{Z}$ biholomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Provide an explicit map $f : \mathbb{C} \rightarrow \mathbb{C}^*$ for the quotient projection.

b. The translation we used in Part (a) is also a biholomorphism of $\mathbb{H}$. In the language of Exercise 2 above it is the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{PSL}(2, \mathbb{R}).$$

Show that the quotient Riemann surface $\mathbb{H}/\mathbb{Z}$ is the punctured disk $\mathbb{D}^* = \mathbb{D}^2 \setminus \{0\}$.

c. **Find** actions of the integers $\mathbb{Z}$ on the upper–half plane $\mathbb{H}$ such that the quotients $\mathbb{H}/\mathbb{Z}$ are the different Riemann surfaces $A_{1,R}$. Note that the parameter $R$ must intervene in the action since $A_{1,R} \not\cong A_{1,S}$ if $R \neq S$.

Express the generator of the action as an element of $\text{PSL}(2, \mathbb{R})$.

d. Let us elaborate a bit more on Part (c) by considering an imaginary number $\alpha \in i\mathbb{R}$ and

$$f_\alpha : \mathbb{H} \rightarrow \mathbb{C}, \quad f(z) = e^{\alpha \ln(z)},$$

which you might be tempted to refer to as $z^\alpha$.

Show that the image of this map $f_\alpha$ is the annulus $A_{\rho,1}$, where $\rho = e^{\alpha \pi i}$.

Composed with the inversion, this serves a covering map

$$\mathbb{H} \rightarrow A_{1,R}$$

for the Riemann surfaces $A_{1,R}$.

e. In the examples above we considered actions on open subsets of $\mathbb{C}$. It is often possible to see symmetries of Riemann surfaces described as algebraic curves by looking at the defining polynomials; a simple example is the following.

Consider Riemann surfaces of the form

$$\{(z, w) \in \mathbb{C}^2 : w^2 = p(z)\}, \quad \deg(p) = 2g + 2.$$

These Riemann surfaces posses the order–two biholomorphism

$$\sigma : (z, w) \mapsto (z, -w).$$

Show that the quotient is a Riemann surface. (Be careful, $\sigma$ has fixed points.)

**Warning**: The part of the problem which asked to describe the symmetry smoothly and computing the quotient requires material from next week; no need to this.

f. Following the idea of finding symmetries from the defining equation, let us consider the following Riemann surface:

$$Q = \{[x : y : z] \in \mathbb{CP}^2 : x^3y + y^3z + z^3x = 0\} \subseteq \mathbb{CP}^2.$$

**Prove** that $Q$ is a smooth Riemann surface.

Show that the biholomorphism group $\text{Aut}(Q)$ contains at least a cyclic group $C_2$ of order 2, a cyclic group $C_3$ of order 3 and a cyclic group $C_7$ of order 7. In particular we see that
the order $|\text{Aut}(Q)|$ is divisible by 42.

**Comment:** This is the famous Klein quartic, which actually appeared in Problem Set 1 as a smooth surface of genus 3. And do not worry, it shall appear again.

**g.** Let us generalize the example in Part (e) and consider the Riemann surfaces

$$K_n = \{ [x, y, z] \in \mathbb{CP}^2 : x y^n + y z^n + z x^n = 0 \}.$$  

**Prove** that $Q$ is a smooth Riemann surface.  

**Show** that the group of biholomorphism contains a cyclic group $C_3$ of order 3 and a cyclic group $C_k$ of order $k = n^2 - n + 1$.

**h.** The last family of Riemann surfaces that we study here are the Fermat curves

$$F_n = \{ [x, y, z] \in \mathbb{CP}^2 : x^n + y^n + z^n = 0 \}.$$  

**Prove** that $Q$ is a smooth Riemann surface.  

**Show** that the group of biholomorphism has at least $6n^2$ elements.

5. **The modular group $PSL(2, \mathbb{Z})$:**

In this exercise we explore the group of integral Möbius transformations $PSL(2, \mathbb{Z})$, which contains many interesting elements and subgroups. This group is contained in the biholomorphisms $PSL(2, \mathbb{R})$ of the upper half plane and inside $PSL(2, \mathbb{C})$, which are the biholomorphisms of the sphere $\mathbb{C} \cup \{\infty\}$. There are no quotients of the latter, and thus one is interested on $PSL(2, \mathbb{Z})$ as a subgroup of $PSL(2, \mathbb{R})$, i.e. as acting on $\mathbb{H}$.

**a.** Geometric intuition for biholomorphisms $PSL(2, \mathbb{R})$ of the upper half plane $\mathbb{H}$ can be trained through the study of explicit examples. Let us consider the following three maps

$$e(z) = -\frac{1}{z}, \quad p(z) = z + 1, \quad h(z) = \frac{z + 1}{z + 2}.$$  

**Describe** these transformations geometrically and **find** their fixed points in $\mathbb{H} \cup \infty$.

It will help your geometric understanding to think about arcs $\gamma \subseteq \mathbb{H}$ with their boundary $\partial \gamma \subseteq \partial \mathbb{H}$ meeting the boundary $\partial \mathbb{H}$ orthogonally: these are the analogues of lines in the plane, and one can think about reflections along such $\gamma$ as the new reflections along a line.

**b.** Let us start looking at the groups these types of transformations generate.

Consider the two groups

$$C_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \langle -\frac{1}{z} \rangle, \quad C_3 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rangle = \langle -\frac{1}{z + 1} \rangle,$$

which are finite cyclic subgroups of $PSL(2, \mathbb{R})$.

**Find** a fundamental domain for $C_2$ and $C_3$.

Every time you have a subgroup of a group of symmetries, finding a fundamental domain is the first step towards understanding the quotient of your space by such subgroup of symmetries.

**c.** Now consider the group generated by both finite order elements studied in Part (b).

This is the group

$$\Gamma = \langle -\frac{1}{z}, -\frac{1}{z + 1} \rangle = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rangle$$

**Show** that $\Gamma = PSL(2, \mathbb{Z})$. Note that in particular this is an infinite group although it is generated by two elements of finite order (which certainly do not commute).
d. Since we have a good description of $\text{PSL}(2,\mathbb{Z})$, we can start to understand the quotient Riemann surface $\mathbb{H}/\text{PSL}(2,\mathbb{Z})$: as before the first step is the fundamental domain.

Find a fundamental domain for the action of $\text{PSL}(2,\mathbb{Z})$ on $\mathbb{H}$.

Describe the smooth surface underlying the Riemann surface $\mathbb{H}/\text{PSL}(2,\mathbb{Z})$.

Extra: Show that the hyperbolic area of the fundamental domain is $\pi/3$.
(I explained this in class, so elaborate the details to some extent.)

This quotient $\mathbb{H}/\text{PSL}(2,\mathbb{Z})$ will feature in the near future, for it will parametrize the cubic algebraic curves from Problem 3 above. Let us however consider new subgroups contained in $\text{PSL}(2,\mathbb{Z})$.

e. Consider the congruence groups

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2,\mathbb{Z}) : a \equiv c \equiv 1, \quad b \equiv d \equiv 0 \right\}.$$  

These feature crucially in the arithmetic study of elliptic curves.

Show that $\Gamma(n)$ is a normal subgroup of $\text{PSL}(2,\mathbb{Z})$ and compute its index for prime $n$.

One of the good things about $\Gamma(n)$ is that it sits in $\text{PSL}(2,\mathbb{Z})$, which we understand a bit due to Part (d). What does this tell us about the fundamental domains of $\Gamma(n)$?

Comment: In the near future we will be able to find a formula for the genus of $\mathbb{H}/\Gamma(N)$.

f. Let us start with $n = 2$ for the $\Gamma(n)$.

Compute the quotient group $\text{PSL}(2,\mathbb{Z})/\Gamma(2)$ and use Part (d) to find a fundamental domain for $\Gamma(2)$. Describe the smooth surface underlying the Riemann surface $\mathbb{H}/\Gamma(2)$.

Extra: Find a fundamental domain for $\Gamma(3)$ and the smooth surface $\mathbb{H}/\Gamma(3)$.

g. (Optional) The group $\text{PSL}(2,\mathbb{Z})$ is not perfect, not every element is the product of commutators. Show that the commutator subgroup of $\text{PSL}(2,\mathbb{Z})$ is generated by

$$[\text{PSL}(2,\mathbb{Z}), \text{PSL}(2,\mathbb{Z})] = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \right\rangle.$$  

In particular the index $[\text{PSL}(2,\mathbb{Z}) : [\text{PSL}(2,\mathbb{Z}), \text{PSL}(2,\mathbb{Z})]]$ is six.

h. (Optional) The reason I am introducing the commutator subgroup is because the quotient surface is different than $\mathbb{H}/\Gamma(2)$ (or $\mathbb{H}/\Gamma(3)$), to start with its symmetries are cyclic whereas $\text{PSL}(2,\mathbb{Z})/\Gamma(2)$ is not even abelian.

Find a fundamental domain for $[\text{PSL}(2,\mathbb{Z}), \text{PSL}(2,\mathbb{Z})]$ and the genus of the smooth surface underlying $\mathbb{H}/[\text{PSL}(2,\mathbb{Z}), \text{PSL}(2,\mathbb{Z})]$.