Abstract. This problem set corresponds to the first week of the course, covering material on the smooth topology of surfaces, and the additional introductory lecture on Sep 7. Choose three of the following five problems, the full grade is computed over these three problems. Every additional problem that you submit will increase the grade.

Advice: If you are not familiar with surfaces, start with Problems 3, 4 and 5. In parts 1.e and 2.e you should express your ideas in terms of the tools you have, in this case graduate students are expected to be able to use modern language.

1. Elliptic Integrals:

   a. Find a function $\sigma_1(x, y)$ such that
      \[ \int_0^x \frac{ds}{\sqrt{1-s^2}} + \int_0^y \frac{ds}{\sqrt{1-s^2}} = \int_0^{\sigma_1(x,y)} \frac{ds}{\sqrt{1-s^2}} \]

   b. Find a function $\sigma_2(x, y)$ such that
      \[ \int_0^x \frac{ds}{\sqrt{1-s^4}} + \int_0^y \frac{ds}{\sqrt{1-s^4}} = \int_0^{\sigma_2(x,y)} \frac{ds}{\sqrt{1-s^4}} \]

   c. Consider the functional identity
      \[ t = \int_0^x \frac{ds}{\sqrt{(1-k^2 s^2)(1-s^2)}} \]
      and define the three functions
      \[ sn(t, k) = x, \quad cn(t, k) = \sqrt{1-x^2}, \quad dn(t, k) = \sqrt{1-k^2 x^2}. \]
      First, prove that these generalize the trigonometric functions:
      \[ \sin(t) = sn(t, 0), \quad \cos(t) = cn(t, 0), \quad 1 = dn(t, 0) \]
      \[ \tanh(t) = sn(t, 1), \quad \sech(t) = cn(t, 1), \quad \sech(t) = dn(t, 1) \]
      Second, show the addition identity:
      \[ sn(t_1 + t_2, k) = \frac{sn(t_1, k)cn(t_2, k)dn(t_2, k) + sn(t_2, k)cn(t_1, k)dn(t_1, k)}{1 - k^2 sn^2(t_1, k) dn^2(t_2, k)} \]

   Hint: we learnt in class that $sn(t, k)$ must satisfy a first degree differential equation, in this case:
   \[(\partial_t sn(t, k))^2 = (1 - sn^2(t, k))(1 - k^2 sn^2(t, k)) \]
   and thus $\partial_t^2 sn(t, k) = -(1 + k^2) sn(t, k) + 2k^2 sn^3(t, k)$). Combine and manipulate this expressions for $sn(t_1, k)$ and $sn(t_2, k)$ until you achieve the right hand side of the addition formula.

   d. Verify the identities $sn^2(t, k) + cn^2(t, k) = 1$ and $k^2 sn^2(t, k) + dn^2(t, k) = 1$. 

e. Suppose that there exist addition identities for \( cn(t_1 + t_2, k) \) and \( dn(t_1 + t_2, k) \), which is correct. Express the information gathered from parts (c) and (d) as geometrically as possible.

2. Quasi–periodicity and periodicity:

a. Starting from a pendulum, give the physical intuition for the double periodicity of the function \( sn(z, k) \) as a function of one complex variable \( z \in \mathbb{C} \).

b. Let \( k \in (0, 1) \), verify the double periodicity
\[
\text{sn}(z, k) = \text{sn}(z + 4\omega_1, k), \quad \text{sn}(z, k) = \text{sn}(z + 2\omega_2, k),
\]
where \( \omega_1 \) and \( \omega_2 \) are the real constants
\[
\omega_1 = \int_0^1 \frac{ds}{\sqrt{(1 - k^2s^2)(1 - s^2)}}, \quad \omega_2 = \int_1^{1/k} \frac{ds}{\sqrt{(1 - k^2s^2)(s^2 - 1)}}
\]

c. Consider the upper–half plane \( \mathbb{H} = \{ z \in \mathbb{C} : \Im (z) > 0 \} \) and the function
\[
\vartheta : \mathbb{C} \times \mathbb{H} \to \mathbb{C}, \quad \vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi in^2 \tau} e^{2\pi inz}.
\]
Verify its periodicity along the real line
\[
\vartheta(z, \tau) = \vartheta(z + 1, \tau)
\]
and its quasi–periodicity along the \( \tau \)–line
\[
\vartheta(z + c\tau, \tau) = e^{-\pi ic^2 \tau} e^{-2\piicz} \vartheta(z, \tau), \quad c \in \mathbb{Z}
\]
d. Consider the functions
\[
\vartheta_{01} = \vartheta(z + 1/2, \tau), \quad \vartheta_{11} = e^{\pi i\tau/4} e^{\pi i(z+1/2)} \vartheta(z + \tau/2 + 1/2, \tau),
\]
use part (c) to conclude that the quotient \( \vartheta_{11}(z, \tau)/\vartheta_{01}(z, \tau) \) is a double periodic function on \( z \in \mathbb{C} \).

This is the point of the exercise: periodicity can be achieved by quotienting quasi–periodic functions. In fact, just for you to know, this is a multiple of our previous friend:
\[
\text{sn}(t, \tau) = -\frac{\vartheta(0, \tau)}{e^{\pi i\tau/4} \vartheta(\tau/2, \tau)} \cdot \frac{\vartheta_{11}(z, \tau)}{\vartheta_{01}(z, \tau)},
\]
where \( t = \pi \vartheta(0, \tau) \cdot z \) and the variable \( k, \tau \) are related via
\[
k = \left( \frac{\vartheta(0, \tau)}{e^{\pi i\tau/4} \vartheta(\tau/2, \tau)} \right)^{-2}.
\]
e. (Optional) Express this exercise in terms of the modern language of algebraic geometry. For instance, \( \text{sn}(z, \tau) \) is a meromorphic function, i.e. a meromorphic section of the trivial line bundle. Interpret then \( \vartheta, \vartheta_{11} \) and \( \vartheta_{01} \) and explain why the quotient is the only reasonable quantity to consider.

Mind–blowing Random Fact: Let \( r_k(n) \) be the number of representations of a natural number \( n \in \mathbb{N} \) by \( k \) squares, then a mild modification of \( \vartheta(0, \tau) \) serves as a generating function of \( r_k(n) \).

\[\text{One way to remember which where the } n \text{ and } n^2 \text{ go in } \vartheta(z, \tau) \text{ is to notice that it solves the heat equation}
\]
\[
\partial_{\tau} \vartheta - \partial_{zz} \vartheta = 0,
\]
which is the heat equation for imaginary time \( i\tau \) and space \( z \in \mathbb{C} \); thus \( \tau \) has the \( n^2 \) and \( z \) the \( n \).
3. **Morse Critical Points:**

Let $f : S \to \mathbb{R}$ be a smooth function on a smooth surface $S$. It is said to be a Morse function if its critical points are non-degenerate.

a. Let $p \in S$ be a non-degenerate critical point of $f : S \to \mathbb{R}$. Then there exist local coordinates $(x_1, x_2)$ in a neighborhood $U \subseteq S$ of the point $p$ such that the function is given by one of the three local models:

$$f(x, y) = x^2 + y^2, \quad f(x, y) = x^2 - y^2, \quad \text{or} \quad f(x, y) = -x^2 - y^2.$$  

Note that this proves that non-degenerate critical points are isolated.

b. Discuss the non-degeneracy of the origin for the smooth functions

$$f(x, y) = \Re(z^3), \quad f(x, y) = x^2, \quad \text{and} \quad \Im(z^2).$$

c. (1-dimensional case) Consider the Taylor expansion

$$\sec t + \tan t = \sum_{n \geq 0} K(n) \frac{t^n}{n!}.$$  

Show that $K(n)$ is the number of pairwise non-equivalent Morse functions $f : \mathbb{R} \to \mathbb{R}$ with $n$ critical points with pairwise different critical values, and asymptotic to $x$ for $n$ even, and $x^2$ for $n$ odd.

The first values are $K(n) = 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936$ for $n = 0, \ldots, 9$. You are allowed to reduce the problem to a combinatorial one, and then claim that the generating function that solves it is indeed $\sec t + \tan t$; in this case provide a reference.

4. **Existence of Morse Functions:**

a. Show that a surface $S$ embeds into some Euclidean space $\mathbb{R}^N$ some $N \in \mathbb{N}$.

b. Give an heuristic argument for the existence of Morse functions on a surface $S$.

(Graduate students should at least invoke the right technical theorems.)

c. Show that there always exists a Morse function with only one index 0 and on index 2 critical points. Find on $S^2$ with 4 critical points of index 1.

Find a Morse function on $\mathbb{RP}^2$. You can either draw it, use homogeneous coordinates $[x_0 : x_1 : x_2] \in \mathbb{RP}^2$ or describe it in your own terms.

d. Find a smooth function on $T^2$ with 3 critical points. Discuss the minimum number of index 1 critical points of a Morse function on $T^2$.

e. Suppose that $f_1 : S_1 \to \mathbb{R}$ and $f_2 : S_2 \to \mathbb{R}$ are Morse functions on two closed surfaces. Construct a Morse function on $S_1 \# S_2$ which relates as much as possible with $f_1$ and $f_2$.

**Random fact (if you already miss elliptic integrals):** L. Nicolaescu has shown in 2 Theorem 6.1 of this article, that the number of Morse functions on the 2–sphere with $k$ saddle points has the inverse of an elliptic integral as generating series.

f. (Optional) How would you complexify the notion of a Morse function ?

For instance, you might want to discuss the function $z \mapsto z^2$ as a complexification of $x \mapsto x^2$. Is there still an accident when crossing the critical value at the origin ?
5. Constructing Surfaces:

a. For each $g \in \mathbb{N}$, find a function $f_g : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the set
$$\Sigma_g = \{(x, y, z) \in \mathbb{R}^3 : f_g(x, y, z) = 0\}.$$  

Note: It is also possible to finding functions whose level sets are non–compact Riemann surfaces. If you are into PDEs, the article [1] shows the existence of harmonic functions whose level sets contain punctured smooth surfaces.

b. Find the genus of the orientable surface obtained by (oppositely) identifying the sides from Figure 1 according to the following identification:
$$1 = 6, \quad 3 = 8, \quad 5 = 10, \quad 7 = 12, \quad 9 = 14, \quad 11 = 2, \quad 13 = 4.$$  

![Figure 1. The quartic of Felix Klein.](image1)

Figure 1. The quartic of Felix Klein.

c. Show that the quotient of the surface $\Sigma_2$ in Figure 2 by the involution $\iota : \Sigma_2 \rightarrow \Sigma_2$ given by a $\pi$–degree rotation along the blue horizontal axis is a smooth surface.

![Figure 2. The hyperelliptic involution.](image2)

Figure 2. The hyperelliptic involution.

d. Find the genus of the quotient surface $\Sigma_2/\iota$ and discuss the number of critical points and values of the quotient projection $\pi : \Sigma_2 \rightarrow \Sigma_2/\iota$.

e. Explain intuitively why any smooth map $f : \Sigma_0 \rightarrow \Sigma_g$ can be deformed to be constant.

f. (Optional) Find the set of homotopy classes of maps from $T^2$ to $\mathbb{RP}^2$.

Compute the set of homotopy classes of maps from $T^2$ to $\Sigma_g$, for all $g \geq 0$. 
References
